Published by Institute of Physics Publishing for SISSA

Received: *June 9, 2008*
Accepted: *August 31, 2008*
Published: *September 11, 2008*

Generating tree amplitudes in $\mathcal{N} = 4$ SYM and $\mathcal{N} = 8$ SG

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Abstract: We study n-point tree amplitudes of $\mathcal{N} = 4$ super Yang-Mills theory and $\mathcal{N} = 8$ supergravity for general configurations of external particles of the two theories. We construct generating functions for n-point MHV and NMHV amplitudes with general external states. Amplitudes derived from them obey SUSY Ward identities, and the generating functions characterize and count amplitudes in the MHV and NMHV sectors. The MHV generating function provides an efficient way to perform the intermediate state helicity sums required to obtain loop amplitudes from trees. The NMHV generating functions rely on the MHV-vertex expansion obtained from recursion relations associated with a 3-line shift of external momenta involving a reference spinor $|X]$. When the shifted amplitude vanishes for large z for all $|X]$, the sum of MHV-vertex diagrams is independent of $|X]$ and gives the correct amplitude. If the shifted amplitude does not vanish for large z, Cauchy's theorem includes a term at infinity. Examples show that special choices of $|X]$ eliminate this term and the MHV vertex expansion becomes valid at these values. We show that the MHV-vertex expansion of the n-graviton NMHV amplitude for $n = 5, 6, \ldots, 11$ is independent of $|X]$ and exhibits the asymptotic behavior z^{n-12}. Generating functions show how the symmetries of supergravity can be implemented in the quadratic map between supergravity and gauge theory embodied in the KLT and other similar relations between amplitudes in the two theories.

Keywords: Supersymmetric gauge theory, Extended Supersymmetry, Supergravity Models.

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1. Introduction

Recent calculations and conjectures [1, 2] on the possible ultraviolet finiteness of $\mathcal{N}=8$ supergravity theory motivate a search for simplifications of the difficult perturbative calculations needed for further progress.[1] Three important techniques used in those calculations are the following:

i. The integrands of loop diagrams are constructed from tree amplitudes using generalized unitarity cuts. Even when external lines are gravitons, the unitarity sum includes processes involving all possible states of the supergravity theory. New information on these tree amplitudes can be helpful at the loop level.

ii. On-shell tree amplitudes in gauge theory and gravity are best expressed using the spinor helicity formalism and are most easily obtained from the modern form of recursion relations [9 − 12] which relate n-point amplitudes to those for smaller values of n. The simplest expressions appear in the MHV sectors of each theory, but perturbative calculations have reached the point where NMHV amplitudes are required. These have been studied for external gluons and gravitons, but there is less information on amplitudes involving other particles of the theory.

iii. Relatively complicated supergravity trees are constructed from the simpler tree amplitudes of $\mathcal{N}=4$ super-Yang-Mills theory using the quadratic relation between gravity and gauge theory embodied in the KLT relations [13]. Implicit in this relation is a map between two copies of the gauge theory and supergravity which we denote by

$$[\mathcal{N}=4\ \mathrm{SYM}]_L \otimes [\mathcal{N}=4\ \mathrm{SYM}]_R \quad \leftrightarrow \quad [\mathcal{N}=8\ \mathrm{SG}]\ . \tag{1.1}$$

There are 16 distinct particle states in each $\mathcal{N}=4$ SYM factor and 256 states in $\mathcal{N}=8$ SG.

This paper is motivated by all three issues above. We focus on the construction of n-point MHV and NMHV tree amplitudes in $\mathcal{N}=4$ SYM and $\mathcal{N}=8$ supergravity with general external states. Toward this end we develop and generalize to supergravity the generating function for MHV amplitudes in gauge theory discovered in [14] and further developed and extended to NMHV amplitudes in [15]. The generating functions encode the external state dependence in a compact way and furnish precise characterizations of the MHV and NMHV sectors. To entice the reader we pose three questions to which the

[1] There are also earlier relevant calculations [3] as well as more recent work [4, 8]. Additional references are given in [1, 2, 4]. The reader is also referred to the reviews [5 − 7].

formalism gives simple answers. The MHV sector of $\mathcal{N} = 4$ SYM consists of the n-gluon amplitude $A_n(- - + + \ldots +)$ with two negative helicity gluons plus all amplitudes related by SUSY transformations. Would the reader have guessed that this sector contains the 8-point amplitude with 8 positive helicity gluini? In supergravity the MHV sector consists of all amplitudes related by SUSY to the n-graviton amplitude $M_n(- - + + \ldots +)$ with two negative helicity gravitons. Would the reader have guessed that there are 186 distinct processes,[2] each with a different set of particles, in this sector? And would the reader have anticipated the external state dependence of n-point MHV amplitudes has a simple direct relationship to the properties of n-point CFT correlators?

Generating functions provide useful answers to a number of questions, and they appear to have practical applications. For example, the unitarity sums over intermediate states required to obtain 1-loop Feynman integrands from the product of tree amplitudes in both gauge theory and supergravity can be done quite efficiently using the generating function.

The generating function for n-point amplitudes in gauge theory is an SU(4) invariant function $F_n(p_i, \eta_{ia})$ of the momenta and of $4n$ Grassmann variables η_{ia}. Here $i = 1, \ldots, n$ refers to the momentum p_i of each external particle and $a = 1, 2, 3, 4$ is the SU(4) flavor index. The generalization to gravity is straightforward in the MHV sector, in which the generating function is an SU(8) invariant function $\Omega(p_i, \eta_{iA})$ of $8n$ Grassmann variables η_{iA} where $A = 1, \ldots, 8$ is an SU(8) index. It is very simple to calculate any MHV amplitude from the generating function by applying Grassmann derivatives specific to the external states. All symmetry transformations can be implemented at the level of the generating function as operations involving the η_{ia} and η_{iA} variables, and one can show that all amplitudes automatically satisfy SUSY Ward identities.

The NMHV sector of gauge theory (or respectively, supergravity) consists of all amplitudes linked by SUSY Ward identities to the n-gluon amplitude $A_n(- - - + + \ldots +)$ (or respectively, the n-graviton $M_n(- - - + \ldots +)$) with 3 negative helicity particles. The construction of a generating function is formally straightforward in the NMHV sector, but its justification is more subtle. There is a different generating function for each diagram in the MHV-vertex expansion of an amplitude. The MHV-vertex expansion was first obtained in ($\mathcal{N} = 0$) gauge theory in [16] and extended to gravity in [17]. The contribution of each diagram depends on the choice of an arbitrary reference spinor $|X]$, but the full amplitude, which is the sum of all diagrams, should be independent of $|X]$.

The simplest justification of the expansion comes from the recursion relation associated with a complex shift of the spinors $|1]$, $|2]$, $|3]$ of the negative helicity lines [18]. The required shift is

$$|m_i] \to |\hat{m}_i] = |m_i] + z\langle m_j\, m_k\rangle |X]\,, \tag{1.2}$$

where m_i, m_j, m_k are the cyclic permutations of the momentum labels for a choice of three of the external lines. For pure gluon or graviton amplitudes, these will be the three negative helicity particles. The recursion relation, and therefore the diagrammatic expansion, is valid if the continued amplitude vanishes as $z \to \infty$. This desired property was proven for gluon amplitudes $A_n(- - - + + \ldots +)$ in [16, 18], but was observed in numerical calculation

[2]For $n < 16$ the number of processes is smaller.

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of the graviton amplitudes $M_n(---+\dots+)$ only for $n=6,7$ in [17]. It is also known for simpler shifts of two external momenta that the large z falloff is slower for amplitudes in which some gluons, or gravitons, are replaced by lower spin particles of the supermultiplet. For these reasons we must be cautious in our applications of the MHV-vertex expansion.

If an amplitude vanishes as $z\to\infty$ for all choices of $|X]$, Cauchy's theorem ensures that the sum of MHV-vertex diagrams is independent of $|X]$. For all NMHV amplitudes in $\mathcal{N}=4$ SYM we show that there is always a choice of 3 lines to shift such that the contribution of each diagram falls at least as fast as $1/z$. We have verified $|X]$-independence of the expansion numerically for a large number of 6-point NMHV amplitudes. Thus we detect no problems, and the generating function appears to be valid for the whole NMHV sector of the $\mathcal{N}=4$ theory.[3]

In gravity the situation is more problematic. For graviton amplitudes $M_n(---+\dots+)$ we show that the falloff as $z\to\infty$ depends on the number of external legs n. Specifically, we have verified numerically for $n=5,\dots,11$ that

$$M_n(\hat{1}^-\,\hat{2}^-\,\hat{3}^-\,4^+\dots n^+)\to\frac{1}{z^{12-n}}\qquad\text{as}\qquad z\to\infty\,. \tag{1.3}$$

This means that for $n\geq 12$, the MHV-vertex decomposition of the n-graviton NHMV amplitude will include a contribution from the residue of the pole at $z=\infty$. Without it, the sum of MHV-vertex diagrams need not be independent of $|X]$. Indeed for $n=12$, there are 1533 diagrams and we find numerically that their sum does depend on $|X]$. As we discuss in other examples below, we expect that it is possible to fix the value of $|X]$ such that the term at infinity vanishes and the MHV-vertex expansion is valid.

The evaluation and summation of diagrams is more complicated for general external states in supergravity so our analysis is limited to 6-point NMHV processes. There are 151 such processes, each with several functionally independent amplitudes obtained by inequivalent assignments of SU(8) indices to the external particles. For each amplitude there are up to 21 non-vanishing diagrams. Most 6-point amplitudes have the same good properties as those of gauge theory; they vanish under large shifts, and they are constructed correctly using the MHV-vertex expansion with diagrams obtained from the generating function.

The large z behavior of individual diagrams for any amplitude can be determined analytically. The result depends on which set of 3 lines are shifted. Our analysis shows that there are processes for which even the best shift contains diagrams with either $O(1)$ or $O(z)$ behavior at large z. Numerical evaluation can then test whether the sum of diagrams depends of $|X]$. This would indicate that the undesired large z behavior persists in the full amplitude, and we have found that it does for a number[4] of examples. In these cases we recalculate the amplitude using the KLT formula which provides a correct evaluation of any $\mathcal{N}=8$ amplitude as a sum of products of $\mathcal{N}=4$ SYM amplitudes. The result from KLT can be continued to complex momenta by shifting spinors and the large z behavior

[3] This is proven in [19].

[4] Of the total of 151 6-point NMHV processes, we estimate that about half will include amplitudes with asymptotic $O(1)$ behavior.

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is then extracted. By this method[5] we have explored about 20 amplitudes whose best shifts give asymptotic $O(1)$ behavior. We call these cases "bad" amplitudes, as opposed to "good" amplitudes which vanish asymptotically for one or more 3-line shifts. The large z limit of these "bad" amplitudes is a ratio of polynomials in the reference spinor $|X]$. The amplitude does not vanish asymptotically for all $|X]$, but it does vanish when $|X]$ is chosen to be a root of the numerator polynomial. The recursion relation becomes valid for these special values of $|X]$, and the sum of MHV-vertex diagrams then agrees with the KLT evaluation. In this way we have developed a good interpretation, and justification, of the generating function even for "bad" amplitudes.

Our analysis also locates two "very bad" 6-point NMHV amplitudes whose KLT evaluations show linear growth in z as $z \to \infty$. Since Cauchy's theorem only picks up the $O(1)$-term at infinity, linear (or faster) growth is not a problem. As above $|X]$ can be chosen to make the $O(1)$-term vanish, and we have checked numerically that the MHV-vertex expansion then agrees with the KLT result.

The need to fix $|X]$ to eliminate a pole at infinity suggests that it may be difficult to apply the generating function to intermediate state helicity sums involving NMHV amplitudes in supergravity. It is important to explore this question, but it is beyond the scope of the present paper.

Let's return to the map (1.1) because another focus of this paper concerns how the $\mathcal{N} = 8$ supersymmetry and global SU(8) symmetry of supergravity are implemented in the tensor product of gauge theory states. One question of concern is how the $\mathrm{SU}(4)_L \otimes \mathrm{SU}(4)_R$ flavor symmetry of the product of gauge theory factors is promoted to the SU(8) global symmetry of supergravity. The derivation of the KLT relations from string theory does not really settle this question, since SU(8) only emerges as an accidental symmetry in the $\alpha' \to 0$ limit.

To investigate such questions we write the detailed algebra of the SUSY charges and the annihilation and creation operators of the gauge and supergravity theories and provide a detailed version of the map (1.1) which is compatible with these symmetry operations. In the map, any SU(8) index $A, B, \ldots \in 1, \ldots, 8$ on the supergravity side splits into $a, b \ldots \in 1, \ldots, 4$ in the left (L) factor of the gauge theory and $r, s \ldots \in 5, \ldots, 8$ in the right (R) factor. Although not manifested in this split, SU(8) transformations can be formally implemented on the gauge theory side of the map of states. We take the attitude that the implementation of SU(8) is better tested on amplitudes, for example through the KLT relations, which read for $n = 4$,

$$M_4(1,2,3,4) = -s_{12}\, A_4(1,2,3,4)_L\, A_4(1,2,4,3)_R\,. \tag{1.4}$$

To apply these to a supergravity process, one places the images of the supergravity operators under the map (1.1) into the gauge factors on the right side of the relations. We will discuss one example, although the notation is not fully described until section 2. Consider the scattering amplitude $\langle b^-(1)\, b^-_{AB}(2)\, b^{CD}_+(3)\, b_+(4)\rangle$ of two gravitons, b^- and b_+,

[5] We have automated the process by writing a Mathematica code which evaluates the KLT expansion as well as the MHV-vertex decomposition for any 6-point NMHV amplitude of the $\mathcal{N} = 8$ theory.

and two graviphotons, b^-_{AB} and b^{CD}_+, with helicities as indicated. The gauge theory images of these operators involve gluons B^-, gluinos F^-_a, and scalars B_{ab}, and the images of the graviphotons depend on whether the SU(8) indices lie in the range $a, b, \ldots \in 1, \ldots, 4$ or $r, s, \ldots \in 5, \ldots, 8$. In other words, the helicity-1 particles can decompose either as $0 \otimes 1$ or as $\frac{1}{2} \otimes \frac{1}{2}$. Using the KLT result (1.4) leads to the formulas

$$\begin{aligned}
\big\langle b^-(1)\, b^-_{ab}(2)\, b^{cd}_+(3)\, b_+(4)\big\rangle &= -s_{12}\, \langle B^-(1)\, B^-_{ab}(2)\, B^{cd}(3)\, B_+(4)\big\rangle_L \\
&\qquad \times \big\langle B^-(1)\, B^-(2)\, B_+(4)\, B_+(3)\rangle_R\,, \qquad (1.5)\\
\big\langle b^-(1)\, b^-_{ar}(2)\, b^{cs}_+(3)\, b_+(4)\big\rangle &= \; s_{12}\, \langle B^-(1)\, F^-_a(2)\, F^c_+(3)\, B_+(4)\big\rangle_L \\
&\qquad \times \big\langle B^-(1)\, F^-_r(2)\, B_+(4)\, F^s_+(3)\rangle_R\,. \qquad (1.6)
\end{aligned}$$

The supergravity amplitude is proportional to the antisymmetric SU(8) tensor δ^{CD}_{AB}, so the product of two bosonic amplitudes in the first expression must equal (to within a sign) the product of fermion amplitudes in the second. This agreement is not a miracle. It must work because the KLT relations are derived from the low energy limit of superstring theory. Nevertheless we are happy to see the sometimes intricate way it does work in this and several other examples we have studied.

The generating function enables us to go beyond examples and give a simple argument that all supergravity symmetries are consistent with the map (1.1). In the MHV sector the supergravity generating function factors into the product of gauge theory generating functions as

$$\Omega_n(p_i, \eta_{iA}) \;\propto\; F_n(p_i, \eta_{ia})_L\, F_n(p_i, \eta_{ir})_R\,. \qquad (1.7)$$

Symmetry transformations of supergravity, written in terms of the η_{iA} variables, automatically work correctly when the η_{iA} split into η_{ia} and η_{ir}, and the transformations applied to the product of gauge theory generating functions on the right side of (1.7). The situation is somewhat more complicated, but very similar in the NMHV sector, where factorization occurs at the level of diagrams.

The plan of the paper is as follows. In section 2 we discuss the algebra of supercharges and the annihilation operators in gauge theory and supergravity and then the operator map. We also discuss the derivation of SUSY Ward identities and their application in the MHV sector. In section 3 we derive the generating functions for the MHV sectors of gauge theory and gravity. An application to the intermediate state helicity sums is presented in section 4. The connection between state dependence of MHV amplitudes and CFT correlators is discussed in section 5. We turn our attention to NMHV amplitudes in section 6. We first discuss recursion relations, especially those derived from (1.2) which lead to the MHV-vertex expansion. Using this we derive the NMHV generating function for gauge theory and discuss its properties. Then we define the NMHV generating function for gravity and discuss the key properties of independence of $|X]$ and behavior as $z \to \infty$. A discussion section concludes the main text. Our conventions are summarized in appendix A. In appendix B, we derive the solution of the SUSY Ward identities for 6-point NMHV $\mathcal{N} = 1$ amplitudes.

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2. SUSY Ward identities and the operator map

In section 2.1 we set up our notation and present the $\mathcal{N} = 4$ and $\mathcal{N} = 8$ SUSY transformation rules for annihilation operators of the bosons and fermions of the gauge and supergravity theories we are concerned with. Further information about our conventions is given in appendix A. In section 2.2 we present the detailed correspondence between the 16×16 products of pairs of gauge theory annihilators and the 256 annihilation operators in supergravity, and in section 2.3 we show how SU(8) transformations can be implemented formally in the product space. We discuss SUSY Ward identities for on-shell amplitudes in $\mathcal{N} = 4$ SYM and $\mathcal{N} = 8$ supergravity in section 2.4. We show by example how to solve the Ward identities in the MHV sectors of the two theories.

2.1 Transformation rules of annihilation operators

We focus on annihilation operators because we adopt the common convention that all particle momenta in an n-point process are viewed as outgoing. An amplitude, such as the n-gluon MHV amplitude, can therefore be viewed as a string of annihilation operators acting to the left on the "out" vacuum. Thus if $B_+(i)$ and $B^-(i)$ are annihilation operators for gluons of energy-momentum p_i^μ and helicity $\pm$, we can represent the color-ordered amplitude as

$$A_n(1^-, 2^-, 3^+, \ldots, n^+) \;=\; \big\langle B^-(1)B^-(2)B_+(3), \ldots, B_+(n)\big\rangle\,. \tag{2.1}$$

In general the amplitudes are regarded as functions of complex null energy-momentum vectors p_i^μ which may be continued to the physical region. If the energy-momentum p^μ is physical, i.e. a positive real null vector, then the operator $B_+(i)$ (or $B^-(i)$) describes a particle in the final state of a physical process. If p^μ is real, but negative null, then the operator corresponds to the anti-particle of opposite helicity in the initial state, which carries physical momentum $-p^\mu$.

The bosons and fermions of $\mathcal{N} = 4$ SYM theory are described by the following annihilators, which are listed in order of descending helicity:

$$B_+(p)\,, \quad F_+^a(p)\,, \quad B^{ab}(p) = \frac{1}{2}\,\alpha_4\,\epsilon^{abcd}B_{cd}(p)\,, \quad F_a^-(p)\,, \quad B^-(p)\,. \tag{2.2}$$

The scalar particles are complex, and satisfy the indicated SU(4) self-duality condition with $\alpha_4 = \pm 1$. The gauge group of the theory is SU(N) with all particles in the adjoint representation. Notation for the "color" degree of freedom is omitted, and we consider only "color-ordered" amplitudes.

The global symmetry group is SU(4), and we use upper and lower indices $a, b, = 1, 2, 3, 4$ to distinguish the two inequivalent conjugate four-dimensional representations. To achieve an SU(4) covariant notation, we separate the left and right chiral components of the $\mathcal{N} = 4$ supercharges and write them as Q_α^a and $\tilde{Q}_a^{\dot\alpha}$ respectively. We then define $Q^a \equiv -\epsilon^\alpha\, Q_\alpha^a$ and $\tilde{Q}_a = \tilde{\epsilon}_{\dot\alpha}\tilde{Q}_a^{\dot\alpha}$, where $\epsilon^\alpha, \tilde{\epsilon}_{\dot\alpha}$ is the anti-commuting parameter of SUSY transformations. (See appendix A for details.) Note that $(\tilde{Q}_a)^\dagger \;=\; Q^a$.

We now state the independent commutation rules for the operators Q^a and $\tilde{Q}_a$ with the various annihilators:

$$\begin{aligned}
\big[\tilde{Q}_a, B_+(p)\big] &= 0\,, & [Q^a, B_+(p)] &= [p\,\epsilon]\, F_+^a(p)\,,\\
\big[\tilde{Q}_a, F_+^b(p)\big] &= \langle\epsilon\, p\rangle\, \delta_a^b\, B_+(p)\,, & \big[Q^a, F_+^b(p)\big] &= [p\,\epsilon]\, B^{ab}(p)\,,\\
\big[\tilde{Q}_a, B^{bc}(p)\big] &= \langle\epsilon\, p\rangle\, \big(\delta_a^b\, F_+^c(p) - \delta_a^c\, F_+^b(p)\big)\,, & \big[Q^a, B^{bc}(p)\big] &= [p\,\epsilon]\, \alpha_4\, \epsilon^{abcd}\, F_d^-(p)\,,\\
\big[\tilde{Q}_a, B_{bc}(p)\big] &= \langle\epsilon\, p\rangle\, \alpha_4\, \epsilon_{abcd}\, F_+^d(p)\,, & \big[Q^a, B_{bc}(p)\big] &= [p\epsilon]\big(\delta_b^a F_c^-(p) - \delta_c^a F_b^-(p)\big)\,,\\
\big[\tilde{Q}_a, F_b^-(p)\big] &= \langle\epsilon\, p\rangle\, B_{ab}(p)\,, & \big[Q^a, F_b^-(p)\big] &= -[p\,\epsilon]\, \delta_b^a\, B^-(p)\,,\\
\big[\tilde{Q}_a, B^-(p)\big] &= -\langle\epsilon\, p\rangle\, F_a^-(p)\,, & \big[Q^a, B^-(p)\big] &= 0\,.
\end{aligned} \tag{2.3}$$

Note that $\tilde{Q}_a$ raises the helicity of all operators and involves the spinor angle bracket $\langle\epsilon\, p\rangle$ in which $|p\rangle \leftrightarrow \lambda_p^{\dot\alpha}$ is the dotted spinor for a particle of momentum p^μ. Similarly, Q^a lowers the helicity and spinor square brackets $[p\,\epsilon]$ appear. Commutators with $B^{bc}(p)$ and $B_{bc}(p)$ are related by self-duality. The Q^a and $\tilde{Q}_a$ operators generate independent Ward identities for n-point amplitudes. We will primarily be concerned with those for $\tilde{Q}_a$.

For distinct SUSY parameters ϵ_1, $\tilde\epsilon_1$ and ϵ_2, $\tilde\epsilon_2$, we define $Q_i^a = -\epsilon_i^\alpha Q_\alpha^a$ and $\tilde{Q}_{ia} = \tilde\epsilon_{i\dot\alpha}\tilde{Q}_{ia}^{\dot\alpha}$. For any operator $\mathcal{O}$ above, the SUSY algebra reads

$$\begin{aligned}
\big[[Q_1^a, \tilde{Q}_{2b}], \mathcal{O}\big] &= \big[Q_1^a, [\tilde{Q}_{2b}, \mathcal{O}]\big] - \big[\tilde{Q}_{2b}, [Q_1^a, \mathcal{O}]\big] = \langle\epsilon_2\, p\rangle[p\,\epsilon_1]\, \delta_b^a\, \mathcal{O}\,,\\
\big[[Q_1^a, Q_2^b], \mathcal{O}\big] &= \big[Q_1^a, [Q_2^b, \mathcal{O}]\big] - \big[Q_2^b, [Q_1^a, \mathcal{O}]\big] = 0\,,\\
\big[[\tilde{Q}_{1a}, \tilde{Q}_{2b}], \mathcal{O}\big] &= \big[\tilde{Q}_{1a}, [\tilde{Q}_{2b}, \mathcal{O}]\big] - \big[\tilde{Q}_{2b}, [\tilde{Q}_{1a}, \mathcal{O}]\big] = 0\,.
\end{aligned} \tag{2.4}$$

Next we proceed in a similar fashion to discuss the transformation rules of $\mathcal{N} = 8$ supergravity, in which the annihilation operators for the 128 bosons and 128 fermions are

$$\begin{aligned}
&b_+(p)\,, \qquad f_+^A(p)\,, \qquad b_+^{AB}(p)\,, \qquad f_+^{ABC}(p)\,,\\
&b^{ABCD}(p) = \frac{1}{4!}\,\alpha_8\, \epsilon^{ABCDEFGH}\, b_{EFGH}(p)\,,\\
&f_{ABC}^-(p)\,, \qquad b_{AB}^-(p)\,, \qquad f_A^-(p)\,, \qquad b^-(p)\,.
\end{aligned} \tag{2.5}$$

The 70 scalars satisfy an SU(8) self-duality condition with $\alpha_8 = \pm 1$. The notation is redundant, since the information on particle type and helicity is determined by the number and position of the global symmetry indices.

There are chiral spinor supercharges Q_α^A and $\tilde{Q}_A^{\dot\alpha}$ which transform in the inequivalent 8 and $\bar{8}$ representations. We contract these charges with a SUSY Grassmann parameter and define $Q^A \equiv -\epsilon^\alpha\, Q_\alpha^A$ and $\tilde{Q}_A \equiv \tilde\epsilon_{\dot\alpha}\tilde{Q}_A^{\dot\alpha}$. It is then straightforward to write SU(8) covariant commutators with annihilation operators:

$$
\begin{aligned}
\big[\tilde{Q}_A,\, b_+\big] &= 0\,, & \big[Q^A,\, b_+\big] &= [p\,\epsilon] f_+^A\,,\\
\big[\tilde{Q}_A,\, f_+^B\big] &= \langle\epsilon\, p\rangle\, \delta_A^B\, b_+\,, & \big[Q^A,\, f_+^B\big] &= [p\,\epsilon]\, b_+^{AB}\,,\\
\big[\tilde{Q}_A,\, b_+^{BC}\big] &= \langle\epsilon\, p\rangle \big(\delta_A^B\, f_+^C - \delta_A^C\, f_+^B\big)\,, & \big[Q^A,\, b_+^{BC}\big] &= [p\,\epsilon]\, f_+^{ABC}\,,\\
\big[\tilde{Q}_A,\, f_+^{BCD}\big] &= \langle\epsilon\, p\rangle \big(\delta_A^B\, b_+^{CD} + \delta_A^C\, b_+^{DB} + \delta_A^D\, b_+^{BC}\big)\,, & \big[Q^A,\, f_+^{BCD}\big] &= [p\,\epsilon]\, b^{ABCD}\,,\\
\big[\tilde{Q}_A,\, b^{BCDE}\big] &= \langle\epsilon\, p\rangle \big(\delta_A^B\, f_+^{CDE} - \delta_A^C\, f_+^{DEB} & \big[Q^A,\, b^{BCDE}\big] &= [p\,\epsilon]\, \tfrac{1}{6}\, \alpha_8\, \epsilon^{ABCDEFGH}\, f_{FGH}^-\,,\\
&\qquad + \delta_A^D\, f_+^{EBC} - \delta_A^E\, f_+^{BCD}\big)\,, & &\\
\big[\tilde{Q}_A,\, b_{BCDE}\big] &= \langle\epsilon\, p\rangle\, \tfrac{1}{6}\, \alpha_8\, \epsilon_{ABCDEFGH}\, f_+^{FGH}\,, & \big[Q^A,\, b_{BCDE}\big] &= [p\,\epsilon] \big(\delta_B^A\, f_{CDE}^- - \delta_C^A\, f_{DEB}^-\\
& & &\qquad + \delta_D^A\, f_{EBC}^- - \delta_E^A\, f_{BCD}^-\big)\,,\\
\big[\tilde{Q}_A,\, f_{BCD}^-\big] &= \langle\epsilon\, p\rangle\, b_{ABCD}\,, & \big[Q^A,\, f_{BCD}^-\big] &= -[p\,\epsilon] \big(\delta_B^A\, b_{CD}^- + \delta_C^A\, b_{DB}^- + \delta_D^A\, b_{BC}^-\big)\,,\\
\big[\tilde{Q}_A,\, b_{BC}^-\big] &= -\langle\epsilon\, p\rangle\, f_{ABC}^-\,, & \big[Q^A,\, b_{BC}^-\big] &= [p\,\epsilon] \big(\delta_B^A\, f_C^- - \delta_C^A\, f_B^-\big)\,,\\
\big[\tilde{Q}_A,\, f_B^-\big] &= \langle\epsilon\, p\rangle\, b_{AB}^-\,, & \big[Q^A,\, f_B^-\big] &= -[p\,\epsilon] \delta_B^A\, b^-\,,\\
\big[\tilde{Q}_A,\, b^-\big] &= -\langle\epsilon\, p\rangle\, f_A^-\,, & \big[Q^A,\, b^-\big] &= 0\,.
\end{aligned}
\tag{2.6}
$$

The supersymmetry generators satisfy (2.4) for any operator $\mathcal{O}$ above.

Supercharge commutators with creation operators can be obtained as the adjoints of the relations given in (2.3) and (2.6). Phases in these commutators have been fixed to be compatible with crossing. Crossing symmetry relates an S-matrix element containing a particle with physical (positive null) momentum in the initial state to the amplitude containing its anti-particle with opposite helicity and unphysical (negative null) momentum in the final state. Thus the SUSY transformation of any creator $a(p, \pm)^*$ must agree with that of the annihilator $a(-p, \mp)$ multiplied by the conventional [20] crossing phase $(-)^{s-\lambda}$ of helicity amplitudes (which has the value -1 only for negative helicity fermions). Note that spinors for negative null momenta satisfy $|-p\rangle = -|p\rangle, \quad |-p] = |p]$.

2.2 The operator map

The precise operator map between $(\mathcal{N}=8) \leftrightarrow (\mathcal{N}=4)_L \otimes (\mathcal{N}=4)_R$ is presented in table 1. Operators in the R gauge theory are dressed with tildes whereas the operators of the L factor are undecorated. The entries in the map are determined, up to signs, by matching the helicity and global symmetry properties of supergravity operators with products of gauge theory operators. Unfixed signs are then determined by compatibility with the scalar self-duality conditions and especially by the consistent action of the supercharges of the $\mathcal{N}=8$ and $\mathcal{N}=4$ theories.

To discuss the implementation of the SUSY commutators we denote a generic annihilation operator by a in supergravity and by A and $\tilde{A}$ in the L and R copies of the gauge theory. The image of any a under the map (1.1) is a specific product $A \otimes \tilde{A}$. A supercharge component Q^a from the first SU(4) sector acts non-trivially only on A, while Q^r from the second sector acts non-trivially only on $\tilde{A}$. Thus we have the scheme

$$\begin{aligned} a(p) &\leftrightarrow A(p)\otimes\tilde{A}(p)\,, \\ [Q^a,a] &\leftrightarrow [Q^a,A\otimes\tilde{A}]\equiv[Q^a,A]\otimes\tilde{A}\,, \\ [Q^r,a] &\leftrightarrow [Q^r,A\otimes\tilde{A}]\equiv A\otimes[Q^r,\tilde{A}]\,, \end{aligned} \tag{2.7}$$

with similar definitions of the action of $\tilde{Q}_a$ and $\tilde{Q}_r$. We then require that the left and right sides of (2.7) still map properly when the transformation rules of section 2.1 are used. This determines the signs of entries in table 1.

Here are two examples, interesting because the two sectors mix. The first example is

$$\Big[Q^a\,,\,b_+^{br}(p)\Big] = [p\,\epsilon]\,f_+^{abr}(p)\,, \tag{2.8}$$

$$\Big[Q^a\,,\,F_+^b(p)\otimes\tilde{F}_+^r(p)\Big] = [p\,\epsilon]\,B(p)^{ab}\otimes\tilde{F}_+^r(p)\,. \tag{2.9}$$

This is compatible with the supersymmetry algebras because the right sides are images under the map (1). The other example is

$$\Big[\tilde{Q}_r\,,\,b^{abcs}(p)\Big] = -\langle\epsilon\,p\rangle\,\delta_r^s\,f_+^{abc}(p)\,, \tag{2.10}$$

$$\Big[\tilde{Q}_r\,,\,F_d^-(p)\otimes\tilde{F}_+^s(p)\Big] = -\langle\epsilon\,p\rangle\,\delta_r^s\,F_d^-(p)\tilde{B}_+(p)\,. \tag{2.11}$$

After multiplication of the second equation by $\alpha_4\,\epsilon^{abcd}$, we see that the map works properly. We have checked explicitly that all entries in the map are consistent with the transformation rules.

There is a choice of the scalar self-duality phases α_8, α_4, and $\tilde{\alpha}_4$ in the $\mathcal{N}=8$ supergravity theory and in the two $N=4$ SYM factors. It turns out that consistency of the map with the commutator algebras requires that

$$\alpha_4\,\tilde{\alpha}_4 = \alpha_8\,. \tag{2.12}$$

We leave α_4, $\tilde{\alpha}_4$, and α_8 arbitrary in the map in table 1, but in applications below we will often set $\alpha_4=\tilde{\alpha}_4=\alpha_8=1$.

2.3 SU(8) symmetry and the operator map

The generators of the fundamental representation of SU(8) are the set of 63 8×8 traceless matrices:

$$(T_B^A)^C{}_D = \delta_D^A\,\delta_B^C - \frac{1}{8}\delta_B^A\,\delta_D^C\,, \tag{2.13}$$

in which A,B denote the Lie algebra element, and C,D are row and column indices. The commutators are

$$[T_B^A\,,\,T_D^C] = \delta_D^A\,T_B^C - \delta_B^C\,T_D^A\,. \tag{2.14}$$

The algebra decomposes with respect to the subgroup $\mathrm{SU}(4)_L\otimes\mathrm{SU}(4)_R\otimes\mathrm{U}(1)$. We use indices ($a,b,\cdots=1,2,3,4$ and $r,s,\cdots=5,6,7,8$). After a minor rearrangement of the

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$b_+ = B_+ \tilde{B}_+$	$b^- = B^- \tilde{B}^-$
$f_+^a = F_+^a \tilde{B}_+$ $f_+^r = B_+ \tilde{F}_+^r$	$f_a^- = F_a^- \tilde{B}^-$ $f_r^- = B^- \tilde{F}_r^-$
$b_+^{ab} = B^{ab} \tilde{B}_+$ $b_+^{ar} = F_+^a \tilde{F}_+^r$ $b_+^{rs} = B_+ \tilde{B}^{rs}$	$b_{ab}^- = B_{ab} \tilde{B}^-$ $b_{ar}^- = -F_a^- \tilde{F}_r^-$ $b_{rs}^- = B^- \tilde{B}_{rs}$
$f_+^{abc} = \alpha_4\, \epsilon^{abcd} F_d^- \tilde{B}_+$ $f_+^{abr} = B^{ab} \tilde{F}_+^r$ $f_+^{ars} = F_+^a \tilde{B}^{rs}$ $f_+^{rst} = \tilde{\alpha}_4\, \epsilon^{rstu} B_+ \tilde{F}_u^-$	$f_{abc}^- = -\alpha_4\, \epsilon_{abcd} F_+^d \tilde{B}^-$ $f_{abr}^- = B_{ab} \tilde{F}_r^-$ $f_{ars}^- = F_a^- \tilde{B}_{rs}$ $f_{rst}^- = -\tilde{\alpha}_4\, \epsilon_{rstu} B^- \tilde{F}_+^u$
$b^{abcd} = \alpha_4\, \epsilon^{abcd} B^- \tilde{B}_+$ $b^{abcr} = \alpha_4\, \epsilon^{abcd} F_d^- \tilde{F}_+^r$ $b^{abrs} = B^{ab} \tilde{B}^{rs}$ $b^{arst} = \tilde{\alpha}_4\, \epsilon^{rstu} F_+^a \tilde{F}_u^-$ $b^{rstu} = \tilde{\alpha}_4\, \epsilon^{rstu} B_+ \tilde{B}^-$	$b_{abcd} = \alpha_4\, \epsilon_{abcd} B_+ \tilde{B}^-$ $b_{abcr} = \alpha_4\, \epsilon_{abcd} F_+^d \tilde{F}_r^-$ $b_{abrs} = B_{ab} \tilde{B}_{rs}$ $b_{arst} = \tilde{\alpha}_4\, \epsilon_{rstu} F_a^- \tilde{F}_+^u$ $b_{rstu} = \tilde{\alpha}_4\, \epsilon_{rstu} B^- \tilde{B}_+$

Table 1: Operator map for annihilators of $(\mathcal{N}=8) \leftrightarrow (\mathcal{N}=4)_L \otimes (\mathcal{N}=4)_R$. Indices $a,b,c,d = (1,2,3,4)$ and $r,s,t,u = (5,6,7,8)$ refer to the splitting of SU(8) into the two separate SU(4) factors.

basis of (2.13), we obtain a set of 63 matrices whose non-vanishing elements are

$$
\begin{aligned}
(T_b^a)^c{}_d &= \delta_d^a\, \delta_b^c - \frac{1}{4}\delta_b^a\, \delta_d^c\,, & (T_s^r)^t{}_u &= \delta_u^r\, \delta_s^t - \frac{1}{4}\delta_s^r\, \delta_u^t\,,\\
(T)^c{}_d &= \delta_d^c\,, & (T)^t{}_u &= -\delta_u^t\,,\\
(T_s^a)^t{}_d &= \delta_d^a \delta_s^t\,, & (T_b^r)^a{}_s &= \delta_s^r \delta_b^a\,.
\end{aligned}
\tag{2.15}
$$

We now define the action of the corresponding Hilbert space operators on the states of the operator map. The generators T_b^a and T_s^r have the usual matrix action of SU(4), defined in (2.15), on gauge theory operators. Nothing special is required. Examples make things clear:

$$
\begin{aligned}
\left[T_b^a\,,\, b_+^{ct}\right] &= \delta_b^c\, b_+^{at}\,, & \left[T_b^a\,,\, F_+^c \otimes \tilde{F}_+^t\right] &= \delta_b^c\, F_+^a \otimes \tilde{F}_+^t\,,\\
\left[T_b^a\,,\, f_c^-\right] &= -\delta_c^a\, f_b^-\,, & \left[T_b^a, F_c^- \otimes \tilde{B}^-\right] &= -\delta_c^a\, F_b^- \otimes \tilde{B}^-\,.
\end{aligned}
\tag{2.16}
$$

The remaining generators are more subtle, but very simple. They have no well defined action on single operators of the gauge theory, but we *define* their action on tensor products of gauge theory operators to match the appropriate supergravity states. The generator T is diagonal on all states. Thus, for example,

$$\big[T\,,\,f^{+abc}\big] \;=\; 3\,f^{+abc}\,, \qquad \big[T\,,\,\alpha_4\epsilon^{abcd}F_d^-\tilde{B}^+\big] \;\equiv\; 3\,\alpha_4\epsilon^{abcd}F_d^-\tilde{B}^+\,. \tag{2.17}$$

For the mixed generators $T_s^a,\ T_b^r$ the definitions require changes from boson to fermion operators in each gauge theory factor. Hence

$$\big[T_b^r\,,\,f^-_{cds}\big] \;=\; -\,\delta_s^r\,f^-_{cdb}\,, \qquad \big[T_b^r\,,\,B_{cd}\otimes\tilde{F}_s^-\big] \;=\; \delta_s^r\,\alpha_4\epsilon_{cdbe}F^{+e}\otimes\tilde{B}^- \tag{2.18}$$

The consistency test for any claimed realization of SU(8) is that the commutation relations (2.14) are satisfied. But this is certainly true here, by explicit construction, since our definitions simply track the conventional implementation of SU(8) in supergravity.

This implementation of SU(8) in the operator map is correct but formal. The acid test is that supergravity amplitudes constructed from gauge theory transform correctly. This requires the kind of non-miracle discussed in the introduction. The dynamical parts of products of very different gauge theory amplitudes must agree, and so must their group theory factors. To show that this non-miracle happens, we will use SUSY Ward identities.

2.4 SUSY Ward identities for on-shell amplitudes

To begin the discussion, we use the generic notation of [21]. An annihilation operator of $\mathcal{N}=4$ SYM or $\mathcal{N}=8$ supergravity is denoted either by α_i or β_i. The subscript i indicates particle momentum, while helicity and global symmetry indices are suppressed. For a pair of supercharges $Q^a,\ \tilde{Q}_a$ of $\mathcal{N}=4$ SYM with fixed SU(4) index, an α operator is defined as one for which $[Q^a,\alpha]$ is non-vanishing, and a β operator is one for which $[\tilde{Q}_a,\beta]$ is non-vanishing. It is clear that $[Q^a,\alpha]=[p\,\epsilon]\,\beta$ and $[\tilde{Q}_a,\beta]=\langle\epsilon\,p\rangle\,\alpha$. The division into α- and β-operators depends on the index choice a. For example, the $\alpha,\ \beta$ operators for the supercharge pair $Q^1,\ \tilde{Q}_1$ are

$$\alpha\text{ operators}: \qquad B_+(p)\,, \qquad F_+^b(p)\,, \qquad B^{bc}(p)\,, \qquad B_{1b}(p)\,, \qquad F_1^-(p)\,. \tag{2.19}$$

$$\beta\text{ operators}: \qquad F_+^1(p)\,, \qquad B^{1b}(p)\,, \qquad B_{bc}\,, \qquad F_b^-(p)\,, \qquad B^-(p)\,, \tag{2.20}$$

where $b,c\neq 1$. The definition of $\alpha,\ \beta$ operators in $\mathcal{N}=8$ supergravity is entirely analogous.

The basic Ward identities are simply the statements that, since supercharges annihilate the vacuum,

$$\begin{aligned} 0 &= \big\langle\big[\tilde{Q}_a,\beta_1\beta_2\ldots\beta_n\,\alpha_{n+1}\alpha_{n+2}\ldots\alpha_{n+m}\big]\big\rangle\,, \\ 0 &= \big\langle\big[Q^a,\beta_1\beta_2\ldots\beta_n\,\alpha_{n+1}\alpha_{n+2}\ldots\alpha_{n+m}\big]\big\rangle\,. \end{aligned} \tag{2.21}$$

By adding and subtracting terms, we convert (2.21) into a sum of commutators $[\tilde{Q}_a,\beta_i]$ or $[Q^a,\alpha_j]$. We can then rewrite (2.21) as

$$0 \;=\; \sum_{i=1}^{n}\langle\epsilon\,i\rangle\,\langle\beta_1\ldots\alpha_i\ldots\beta_n\,\alpha_{n+1}\ldots\alpha_{n+m}\rangle\,, \tag{2.22}$$

$$0 \;=\; \sum_{j=n+1}^{n+m}[j\,\epsilon]\,\langle\beta_1\ldots\beta_n\,\alpha_{n+1}\ldots\beta_j\ldots\alpha_{n+m}\rangle\,. \tag{2.23}$$

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Since the spinors have two components, the analytic and anti-analytic expressions each contain two independent constraints on the amplitudes. To obtain a useful identity one must start with a string of operators in (2.21) which contains an odd number of fermion annihilators. Then the individual amplitudes which appear in the constraints contain an even number of fermions. Otherwise they vanish trivially. The ordering of operators is relevant in gauge theory because amplitudes are color ordered, but it has no significance in supergravity.

Let's consider the two cases in which the initial string of operators in (2.21) contains only one or two α operators, respectively. Then the Q^a Ward identities read

$$[(n+1)\,\epsilon]\,\langle\beta_1\ldots\beta_n\beta_{n+1}\rangle = 0\,, \tag{2.24}$$

$$[(n+1)\,\epsilon]\,\langle\beta_1\ldots\beta_n\alpha_{n+1}\beta_{n+2}\rangle \;+\; [(n+2)\,\epsilon]\,\langle\beta_1\ldots\beta_n\beta_{n+1}\alpha_{n+2}\rangle = 0\,. \tag{2.25}$$

We now exploit the freedom to choose the two-component spinor ϵ_α. We can choose it so that $[(n+1)\,\epsilon] \neq 0$. Then (2.24) tells us that any amplitude which contains only β operators must vanish. To exploit the information in (2.25) we choose, in turn, $|\epsilon] \sim |n+2]$ and then $|\epsilon] \sim |n+1]$. We learn that any amplitude with n β operators and one α operator must vanish. By similar arguments, we can use the $\tilde{Q}_a$ Ward identity to show that any amplitude containing at most one β operator must vanish. These statements comprise the well known helicity conservation rules for n-point functions. For amplitudes containing only gluons, they read $A_n(1^+,2^+,\ldots,n^+) = 0$ and $A_n(1^+,\ldots,(n-1)^+,n^-) = 0$.

Relations between different amplitudes are obtained when the initial string contains $k \geq 3$ β operators plus $n-k \geq 1$ α operators. The case of exactly three β operators is particularly easy to analyze and very useful. The analytic Ward identity reads

$$\langle\epsilon\,1\rangle\,\langle\alpha_1\beta_2\beta_3\alpha_4\ldots\alpha_n\rangle \;+\; \langle\epsilon\,2\rangle\,\langle\beta_1\alpha_2\beta_3\alpha_4\ldots\alpha_n\rangle \;+\; \langle\epsilon\,3\rangle\,\langle\beta_1\beta_2\alpha_3\alpha_4\ldots\alpha_n\rangle \;=\; 0\,. \tag{2.26}$$

As stated above this equation contains two independent relations among the three amplitudes involved. By choosing $|\epsilon\rangle = |2\rangle$ and then $|\epsilon\rangle = |1\rangle$, we obtain

$$\langle\alpha_1\beta_2\beta_3\alpha_4\ldots\alpha_n\rangle \;=\; -\frac{\langle 2\,3\rangle}{\langle 2\,1\rangle}\langle\beta_1\beta_2\alpha_3\alpha_4\ldots\alpha_n\rangle\,, \tag{2.27}$$

$$\langle\beta_1\alpha_2\beta_3\alpha_4\ldots\alpha_n\rangle \;=\; -\frac{\langle 1\,3\rangle}{\langle 1\,2\rangle}\langle\beta_1\beta_2\alpha_3\alpha_4\ldots\alpha_n\rangle\,. \tag{2.28}$$

An example of these relations is the $\big\langle[\tilde{Q}_a, B^-(1)B^-(2)F^b_+(3)B_+(4)\ldots B_+(n)]\big\rangle \;=\; 0$ Ward identity in gauge theory. The two constraints above then become

$$\big\langle F^-_a(1)B^-(2)F^b_+(3)B_+(4)\ldots B_+(n)\big\rangle \;=\; \delta^b_a\frac{\langle 2\,3\rangle}{\langle 2\,1\rangle}\big\langle B^-(1)B^-(2)B_+(3)\ldots B_+(n)\big\rangle\,, \tag{2.29}$$

$$\big\langle B^-(1)F^-_a(2)F^b_+(3)B_+(4)\ldots B_+(n)\big\rangle \;=\; \delta^b_a\frac{\langle 1\,3\rangle}{\langle 1\,2\rangle}\big\langle B^-(1)B^-(2)B_+(3)\ldots B_+(n)\big\rangle\,. \tag{2.30}$$

Thus an amplitude containing a pair of opposite helicity gluinos is related to the well known MHV n-gluon amplitude. For this reason the set of amplitudes with two β operators and

$n-2$ α operators is called the MHV sector of the theory. Note that the gluinos can be placed in any positions by change in the placement of the three initial β operators.

As another example of an MHV Ward identity in the gauge theory, consider

$$\big\langle [\tilde{Q}_a, B^-(1) F_b^-(2) B^{cd}(3)\, B_+(4) \dots B_+(n)] \big\rangle = 0\,,$$

and use (2.30) to simplify the $\langle \epsilon\, 3\rangle$ terms. With $|\epsilon\rangle \sim |1\rangle$ or $\sim |2\rangle$ we get two Ward identities:

$$\big\langle B^-(1)\, B_{ab}(2)\, B^{cd}(3)\, B_+(4) \dots B_+(n) \big\rangle \;=\; 2\delta^{cd}_{ab}\, \frac{\langle 13\rangle^2}{\langle 12\rangle^2}\, \big\langle B^-(1)\, B^-(2)\, B_+(3) \dots B_+(n) \big\rangle\,, \tag{2.31}$$

$$\big\langle F^-_a(1)\, F^-_b(2)\, B^{cd}(3)\, B_+(4) \dots B_+(n) \big\rangle \;=\; 2\delta^{cd}_{ab}\, \frac{\langle 13\rangle\langle 23\rangle}{\langle 12\rangle^2}\, \big\langle B^-(1)\, B^-(2)\, B_+(3) \dots B_+(n) \big\rangle\,. \tag{2.32}$$

Anti-symmetrizers are defined as $\delta^{b_1 \dots b_n}_{a_1 \dots a_n} = \frac{1}{n!}(\delta^{b_1}_{a_1} \cdots \delta^{b_n}_{a_n} \pm \text{perms})$. It is also easy to derive, see [22],

$$\big\langle B_+(1) \dots B^-(i) \dots B^-(j) \dots B_+(n) \big\rangle \;=\; \frac{\langle i\, j\rangle^4}{\langle 1\, 2\rangle^4}\, \big\langle B^-(1)\, B^-(2)\, B_+(3) \dots B_+(n) \big\rangle\,. \tag{2.33}$$

Let's examine the anti-analytic $\big\langle [Q^b, B^-(1)\, F_a^-(2) B_+(3) \dots B_+(n)] \big\rangle = 0$ Ward identity, which gives the relation

$$\begin{aligned} &[2\, \epsilon]\delta^b_a \big\langle B^-(1) B^-(2) B_+(3) \dots B_+(n) \big\rangle \\ &\qquad + \sum_{j=3}^{n} [j\, \epsilon] \big\langle B^-(1) F_a^-(2) B_+(3) \dots F_+^b(j) \dots B_+(n) \big\rangle \;=\; 0\,. \end{aligned} \tag{2.34}$$

If we use the previous result (2.30) and its extension to the case where $F_+^a(j)$ appears, then (2.34) reduces to

$$\Big(\sum_{j=2}^{n} \langle 1\, j\rangle [j\, \epsilon] \Big) \delta^b_a \big\langle B^-(1) B^-(2) B_+(3) \dots B_+(n) \big\rangle = 0\,. \tag{2.35}$$

But the sum of products of spinor brackets vanishes because of momentum conservation, so the anti-analytic Ward identity (2.34) is satisfied, after information from the analytic Ward identities is used. This is a general feature of the MHV sector, but it is not true in the NMHV sector and beyond.

Ward identities for amplitudes in the MHV sector of $\mathcal{N} = 8$ supergravity can be obtained in a similar fashion. With appropriate choices of the α, β operators (and of $|\epsilon\rangle$)

one can derive the useful results:

$$
\begin{aligned}
&\big\langle b^-(1) f^-_A(2)\, b_+(3) \ldots f^B_+(k) \ldots b_+(n)\big\rangle \\
&\qquad\qquad = \delta^B_A\, \frac{\langle 1k\rangle}{\langle 12\rangle} \big\langle b^-(1) b^-(2) b_+(3) \ldots b_+(n)\big\rangle\,, \\
&\big\langle b^-(1) b^-_{AB}(2)\, b_+(3) \ldots b^{CD}_+(k) \ldots b_+(n)\big\rangle \\
&\qquad\qquad = 2\, \delta^{CD}_{AB}\, \frac{\langle 1k\rangle^2}{\langle 12\rangle^2} \big\langle b^-(1) b^-(2) b_+(3) \ldots b_+(n)\big\rangle\,, \\
&\big\langle b^-(1) f^-_{ABC}(2)\, b_+(3) \ldots f^{DEF}_+(k) \ldots b_+(n)\big\rangle \\
&\qquad\qquad = 3!\, \delta^{DEF}_{ABC}\, \frac{\langle 1k\rangle^3}{\langle 12\rangle^3} \big\langle b^-(1) b^-(2) b_+(3) \ldots b_+(n)\big\rangle\,, \\
&\big\langle b^-(1) b_{ABCD}(2)\, b_+(3) \ldots b^{EFGH}(k) \ldots b_+(n)\big\rangle \\
&\qquad\qquad = 4!\, \delta^{EFGH}_{ABCD}\, \frac{\langle 1k\rangle^4}{\langle 12\rangle^4} \big\langle b^-(1) b^-(2) b_+(3) \ldots b_+(n)\big\rangle\,.
\end{aligned} \tag{2.36}
$$

It is easy to write generic Ward identities in the NMHV sector, but much harder to extract useful information from them. From $\big\langle [\tilde{Q}_a\,,\, \beta_1\beta_2\beta_3\beta_4\alpha_5 \ldots \alpha_n]\big\rangle = 0$, one derives

$$
\langle \epsilon\, 1\rangle\, f_1 + \langle \epsilon\, 2\rangle\, f_2 + \langle \epsilon\, 3\rangle\, f_3 + \langle \epsilon\, 4\rangle\, f_4 \;=\; 0\,, \tag{2.37}
$$

with $f_1 = \langle \alpha_1\beta_2\beta_3\beta_4\alpha_5 \ldots \alpha_n\rangle$, $f_2 = \langle \beta_1\alpha_2\beta_3\beta_4\alpha_5 \ldots \alpha_n\rangle$, $f_3 = \langle \beta_1\beta_2\alpha_3\beta_4\alpha_5 \ldots \alpha_n\rangle$, and $f_4 = \langle \beta_1\beta_2\beta_3\alpha_4\alpha_5 \ldots \alpha_n\rangle$. By choice of ϵ one can derive two independent relations among the four amplitudes. Given one set of amplitudes f_i which satisfy (2.37), then one may use the Schouten identity to show that another one is given by $f_1 + \langle 2\,3\rangle f_0,\;\; f_2 + \langle 3\,1\rangle f_0,\;\; f_3 + \langle 1\,2\rangle f_0,\;\; f_4$, where f_0 is an arbitrary function. Thus additional information is required to specify the amplitudes [21]. The solution for $\mathcal{N} = 1$ 6-point functions in [21] is rederived by spinor-helicity methods in appendix B. It could be very useful to develop techniques to solve the NMHV Ward identities, particularly for extended SUSY.

Many of the properties we have illustrated above in the examples are neatly encoded in generating functions for MHV amplitudes. This is our next subject.

3. Generating functions for MHV amplitudes

In section 2.4 we showed that SUSY Ward identities are quite simple in the MHV sectors of $\mathcal{N} = 4$ SYM and $\mathcal{N} = 8$ supergravity, indeed amenable to step-by-step solution. Nevertheless, a systematic method of solution for the entire MHV sector is awkward at best. Nor do we yet know a simple way to determine whether particular amplitudes, such as $\langle B^-(1)\, F^a_+(2)\, F^b_+(3)\, F^c_+(4)\, F^d_+(5)\, B_+(6) \ldots B_+(n)\rangle$ or the 8-gluino amplitude mentioned in the Introduction are within the MHV sector. The remarkable generating function derived for the gauge theory by Nair [14], and further developed by Georgiou, Glover, and Khoze [15], provides very simple answers to these questions. In this section we explain and elucidate new properties of this generating function and generalize it to the MHV sector of $\mathcal{N} = 8$ supergravity. Then we show that it embodies a clear explanation of the compatibility of $\mathcal{N} = 8$ SUSY and SU(8) global symmetry with the map (1.1).

3.1 Gauge theory

Suppose that we are interested in the full sector of MHV n-point functions in the gauge theory. Following Nair, we introduce a set of $4n$ anti-commuting variables η_{ia} in which $i = 1, \ldots, n$ indicates particle momentum, and $a = 1, 2, 3, 4$ is the SU(4) index. The generating function depends on the η_{ia} and the (commuting) spinors $\tilde{\lambda}_i^{\dot{\alpha}} \leftrightarrow |i\rangle$ which encode particle momenta. The generating function is

$$F_n = \left(\prod_{i=1}^{n} \langle i\,(i+1)\rangle \right)^{-1} \delta^{(8)}\left(\sum_{i=1}^{n} |i\rangle \eta_{ia} \right), \tag{3.1}$$

and the 8-dimensional δ-function can be expressed as

$$\delta^{(8)}\left(\sum_{i=1}^{n} |i\rangle \eta_{ia} \right) = \frac{1}{16} \prod_{a=1}^{4} \sum_{i,j=1}^{n} \langle i\, j\rangle\, \eta_{ia}\, \eta_{ja}\,. \tag{3.2}$$

It is a sum of $\left(\frac{1}{2}n(n-1)\right)^4$ terms, each involving a product of 8 distinct η_{ia}; it is invariant under SU(4) transformations of the η_{ai} and under cyclic permutations of the momentum labels i.

The coefficient of each distinct product of 8 η_{ai} is an MHV amplitude when interpreted by means of the prescription of [15]. We restate this prescription in terms of products of derivatives. Each annihilation operator of the gauge theory is associated with a differential operator[6] as follows:

$$\begin{aligned}
&B_+(i) \leftrightarrow 1, && F_+^a(i) \leftrightarrow D_i^a \frac{\partial}{\partial \eta_{ia}}\,, \\
&B^{ab}(i) \leftrightarrow D_i^{ab} = \frac{\partial^2}{\partial\eta_{ia}\partial\eta_{ib}}\,, && B_{ab}(i) \leftrightarrow D_{iab} = \frac{1}{2}\epsilon_{abcd} D_i^{cd}\,, \\
&F_a^-(i) \leftrightarrow D_{ia} = -\frac{1}{6}\epsilon_{abcd}\frac{\partial^3}{\partial\eta_{ib}\partial\eta_{ic}\partial\eta_{id}}\,, && B^-(i) \leftrightarrow D_i = \frac{1}{24}\epsilon_{abcd}\frac{\partial^4}{\partial\eta_{ia}\partial\eta_{ib}\partial\eta_{ic}\partial\eta_{id}}\,.
\end{aligned} \tag{3.3}$$

Any desired MHV amplitude is obtained by applying an 8th order differential operator composed as the product of appropriate factors from (3.3). For example, the n-gluon Parke-Taylor [23] amplitude is given by

$$A_n(1^-, 2^-, 3^+, \ldots, n^+) = D_1\, D_2\, F_n = \frac{\langle 1\,2\rangle^4}{\prod_{i=1}^{n} \langle i\,(i+1)\rangle}\,. \tag{3.4}$$

We can use this to write the generating function in the alternate form

$$F_n = \frac{A_n(1^-, 2^-, 3^+, \ldots, n^+)}{\langle 1\,2\rangle^4} \delta^{(8)}\left(\sum_{i=1}^{n} |i\rangle \eta_{ia} \right), \tag{3.5}$$

[6] The product structure of F_n suggests that we use upper indices for all fields, thus $(F^-)^{abc}(i) \leftrightarrow \frac{\partial^3}{\partial\eta_{ia}\partial\eta_{ib}\partial\eta_{ic}}$. The lower index field is then defined as the dual, i.e. $(F^-)^{abc}(i) = \epsilon^{abcd} F_d^-(i)$. This definition leads to the $-$ sign in the derivative D_{ia}. Similar remarks apply to the negative chirality fields in supergravity and the associated differential operators. See (3.15) below.

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which is useful to compare with extensions discussed below.

Any product of 8 derivatives produces an amplitude in the MHV sector of the gauge theory. Since the maximum order of any individual operator is 4, each 8th order differential operator is associated with a partition of the integer 8 with maximum summand $n_{\max} \leq 4$. Each partition corresponds to a particular set of particles in an n-point MHV amplitude. There are 15 such partitions, which correspond to the 15 types of MHV amplitude listed in (5.4) of [15]. For example, the $\langle B^-(1)\, F^a_+(2)\, F^b_+(3)\, F^c_+(4)\, F^d_+(5)\, B_+(6) \ldots B_+(n)\rangle$ amplitude mentioned in the first paragraph of this section corresponds to the partition $8 = 4+1+1+1+1$, and the 8-gluino amplitude of the Introduction is $8 = 1+1+1+1+1+1+1+1$.

How do we know that all amplitudes obtained by differentiation of F_n agree with those produced by explicit stepwise solution of the Ward identities? To answer this question we show below that the amplitudes obtained from F_n satisfy the SUSY Ward identities, and we already observed above that the n-gluon Parke-Taylor amplitude is correctly produced. The solution of the MHV Ward identities is unique under these conditions, so the favorable conclusion is valid.

We define supercharges

$$\tilde{Q}_a \;=\; \sum_{i=1}^{n} |i\rangle\, \eta_{ia}\,, \qquad Q^a \;=\; \sum_{i=1}^{n} [\, i\,|\, \frac{\partial}{\partial \eta_{ia}}\,, \tag{3.6}$$

which act by multiplication and differentiation in the space of functions of the η's. Their anticommutator is

$$\{Q^a\,,\, \tilde{Q}_b\} \;=\; \delta^a_b \sum_{i=1}^{n} |\, i\rangle\, [i\,| \;=\; 0\,. \tag{3.7}$$

The fact that it vanishes due to momentum conservation should not be a surprise, since (3.7) corresponds exactly to the basic SUSY anticommutator $\{Q^{a\alpha}\,,\, \tilde{Q}^{\dot\beta}_b\} = \delta^a_b\, P^{\dot\beta\alpha}$ which also vanishes when the operator $P^{\dot\beta\alpha}$ is applied to an amplitude.

Consider the spinor contraction $\langle \epsilon\, \tilde{Q}_a\rangle = \sum_i \langle \epsilon\, i\rangle \eta_{ia}$ of the supercharge $\tilde{Q}_a$ in (3.6) with the parameter ϵ. The set of commutators of this operator with the differential operators of (3.3) is isomorphic to the commutator algebra of (2.3). For example,

$$\begin{aligned}
\Big[\langle \epsilon\, \tilde{Q}_a\rangle\,,\, 1\Big] &= 0\,,\\
\Big[\langle \epsilon\, \tilde{Q}_a\rangle\,,\, D^b_i\Big] &= \langle \epsilon\, i\rangle\, \delta^b_a\, 1\,,\\
\Big[\langle \epsilon\, \tilde{Q}_a\rangle\,,\, D^{bc}_i\Big] &= \langle \epsilon\, i\rangle\, \big(\delta^b_a\, D^c_i - \delta^c_a\, D^b_i\big)\,, \qquad \text{etc.}
\end{aligned} \tag{3.8}$$

Thus the correspondence (3.3) between particle annihilators and differential operators respects SUSY.

It may seem that there is at most a half-truth here since the commutators of $[Q^a\, \epsilon] = \sum_i [i\, \epsilon] \partial/\partial\, \eta_{ia}$ with all operators of (3.3) *vanish* rather than mirror the structure of (2.3). This apparent paradox requires more thought. It may be related to the fact that the Q^a Ward identities are automatically satisfied in the MHV sector and are thus suggestive of a type of 1/2-BPS property which we discuss further in section 7.

The SUSY Ward identities hold formally in the form

$$\tilde{Q}_a F_n \;=\; 0\,, \qquad\qquad Q^a F_n \;=\; 0\,, \tag{3.9}$$

the first because we are multiplying $\delta^{(8)}$ by its own argument and the second by momentum conservation. We view these formal Ward identities as the analogue of the statement (2.21). The concrete Ward identities of section 2.4 are obtained from products of the form

$$D^{(9)}\left(\langle\epsilon\,\tilde{Q}_a\rangle F_n\right) \;=\; 0\,, \tag{3.10}$$

where $D^{(9)}$ is a product of operators from the correspondence (3.3) of total order 9. Similarly explicit Ward identities of the supercharge Q^a are obtained from products of the form

$$[\epsilon\,Q^a]\,D^{(7)}F_n \;=\; 0\,. \tag{3.11}$$

This expression is a sum of 8th order derivatives. There are two possibilities depending on the SU(4) indices of the product operator $D^{(7)}$. Either each individual term vanishes due to SU(4) symmetry, or there are three non-vanishing terms[7] which constitute an explicit Q^a Ward identity relating three amplitudes. These comments about SU(4) symmetry also apply to the $\tilde{Q}_a$ Ward identities of (3.10).

3.2 Practicalities

As an example will show, the computation of spin factors from $D^{(8)}\delta^{(8)}\big(\sum_{i=1}^n |i\rangle\eta_{ia}\big)$, reduces to a simple Wick contraction algorithm of the basic operators $\partial_i^a \equiv \partial/\partial\eta_{ia}$. The elementary contraction is

$$\hat{\partial}_i^a \ldots \hat{\partial}_j^b \;=\; \pm\delta^{ab}\langle i\,j\rangle\,. \tag{3.12}$$

The $\ldots$ indicates other operators between those which are contracted and the sign depends on whether the number of these operators is even or odd. Suppose we want to obtain the amplitude of (2.32) for the specific index values $a = c = 1$, $b = d = 2$. From the correspondence (3.3) we see that we must compute

$$\begin{aligned} D^{(8)}\delta^{(8)}\left(\sum_{i=1}^n |i\rangle\eta_{ia}\right) &= -\partial_1^2\partial_1^3\partial_1^4\,\partial_2^1\partial_2^3\partial_2^4\,\partial_3^1\partial_3^2\,\frac{1}{16}\prod_{a=1}^4\sum_{i,j=1}^n\langle i\,j\rangle\,\eta_{ia}\,\eta_{ja} \\ &= \langle 1\,2\rangle^2\langle 2\,3\rangle\langle 1\,3\rangle\,. \end{aligned} \tag{3.13}$$

The spin factor obtained by explicit action on $\delta^{(8)}$ is more easily found by pairwise Wick contraction of the operators in the string $D^{(8)}$. When the spin factor in (3.13) is multiplied by the dynamical prefactor in the generating function (3.1), one finds exactly the amplitude produced by explicit solution of the Ward identities in (2.32).

[7] In some cases more than three terms appear, but all except three vanish when definite values are assigned to the SU(4) indices.

3.3 Gravity

The good news now is that it is a very straightforward matter to write down a generating function for the MHV sector of $\mathcal{N} = 8$ supergravity. To describe n-point functions one now needs $8n$ anti-commuting variables η_{iA} in which A is an SU(8) index. The generating function is then

$$\Omega_n = \frac{M_n(1^-, 2^-, 3^+, \ldots n^+)}{\langle 1\,2\rangle^8}\,\delta^{(16)}\Big(\sum_{i=1}^{n} |i\rangle \eta_{iA}\Big)\,, \qquad (3.14)$$

$$\text{with} \quad \delta^{(16)}\Big(\sum_{i=1}^{n} |i\rangle \eta_{iA}\Big) = \frac{1}{256}\prod_{A=1}^{8}\sum_{i,j=1}^{n} \langle i\, j\rangle\, \eta_{iA}\,\eta_{jA}\,.$$

The quantity $M_n(1^-, 2^-, 3^+, \ldots, n^+)$ is the n-graviton MHV amplitude which can be written using the KLT relations [13] or one of the several specific forms available for MHV amplitudes [24, 8, 25]. Although it is not obvious, the quantity $M_n(1^-, 2^-, 3^+, \ldots n^+)/\langle 1\,2\rangle^8$ is invariant under the exchange $i \leftrightarrow j$ of *any* pair of lines. This property actually follows from the SUSY Ward identities [22]. Thus the formula (3.14) is entirely Bose symmetric. It is also SU(8) invariant. It is a sum of products of 16 distinct η's.

To use the generating function Ω_n we define a new set of differential operators:

$$\begin{aligned}
\mathcal{D}_i^A &= \frac{\partial}{\partial \eta_{iA}}\,, & \mathcal{D}_i^{AB} &= \frac{\partial^2}{\partial \eta_{iA}\partial \eta_{iB}}\,, \\
\mathcal{D}_i^{ABC} &= \frac{\partial^3}{\partial \eta_{iA}\partial \eta_{iB}\partial \eta_{iC}}\,, & \mathcal{D}_i^{ABCD} &= \frac{\partial^4}{\partial \eta_{iA}\partial \eta_{iB}\partial \eta_{iC}\partial \eta_{iD}}\,, \\
\mathcal{D}_{iABC} &= -\frac{1}{5!}\epsilon_{ABCDEFGH}\frac{\partial^5}{\partial \eta_{iD}\cdots\partial \eta_{iH}}\,, & \mathcal{D}_{iAB} &= \frac{1}{6!}\epsilon_{ABCDEFGH}\frac{\partial^6}{\partial \eta_{iC}\cdots\partial \eta_{iH}}\,, \\
\mathcal{D}_{iA} &= -\frac{1}{7!}\epsilon_{ABCDEFGH}\frac{\partial^7}{\partial \eta_{iB}\cdots\partial \eta_{iH}}\,, & \mathcal{D}_{i} &= \frac{1}{8!}\epsilon_{ABCDEFGH}\frac{\partial^8}{\partial \eta_{iA}\cdots\partial \eta_{iH}}\,.
\end{aligned} \qquad (3.15)$$

As in the case of gauge theory, the fields of supergravity are associated with these operators as follows:

$$\begin{aligned}
&b_+(i) \leftrightarrow 1\,, & &f_+^A(i) \leftrightarrow \mathcal{D}_i^A\,, & &b_+^{AB}(i) \leftrightarrow \mathcal{D}_i^{AB}\,, \\
&f_+^{ABC}(i) \leftrightarrow \mathcal{D}_i^{ABC}\,, & &b^{ABCD}(i) \leftrightarrow \mathcal{D}_i^{ABCD}\,, & &f_{ABC}^-(i) \leftrightarrow \mathcal{D}_{iABC}\,, \\
&b_{AB}^-(i) \leftrightarrow \mathcal{D}_{iAB}\,, & &f_A^-(i) \leftrightarrow \mathcal{D}_{iA}\,, & &b^-(i) \leftrightarrow \mathcal{D}_i\,.
\end{aligned} \qquad (3.16)$$

To obtain the MHV amplitude for a particular set of external lines one simply applies a 16th order differential operator which is the product of the corresponding operators from (3.16). As a typical example, we write

$$\begin{aligned}
\big\langle b^-(1)\, b_{AB}(2)\, b^{CD}(3)\, b_+(4)\ldots b_+(n)\big\rangle &= \mathcal{D}_1\, \mathcal{D}_{2AB}\, \mathcal{D}_3^{CD}\, \Omega_n \\
&= 2\,\delta_{AB}^{CD}\,\frac{\langle 1\,3\rangle^2}{\langle 1\,2\rangle^2}\, M_n(1^-, 2^-, 3^+, \ldots, n^+)\,, \qquad (3.17)
\end{aligned}$$

which agrees with (2.36).

It is significant that the state dependent spinor factors obtained from $\delta^{(16)}$ involve only analytic spinor brackets $\langle i\, j\rangle$, although complete supergravity amplitudes also involve anti-analytic spinor brackets $[i\, j]$.

It should be clear that any product of the derivatives in (3.15) of order 16 produces an amplitude in the MHV sector, and that we can associate a partition of 16 with $n_{\max} \leq 8$ with each distinct product. There are 186 such partitions, each of which corresponds to an n-point amplitude for a particular set of external fields. For example the amplitude in (3.17) corresponds to the partition $16 = 8 + 6 + 2$.

It is also clear from the preceding discussion in gauge theory that the amplitudes generated in this way satisfy the SUSY Ward identities for $\mathcal{N} = 8$ supergravity. Since these Ward identities have a unique solution in the MHV sector, the amplitudes so constructed are correct. Each of the 186 MHV amplitudes is the product of the n-graviton amplitude times a state-dependent spin factor which is a homogeneous function with $k \leq 8$ angle bracket factors $\langle i\, j\rangle$ in the numerator and $\langle 1\, 2\rangle^k$ in the denominator.

We now put readers on notice that the punch line of our argument concerning the realization of SU(8) global symmetry in the map from gauge theory to supergravity is near, at least for MHV amplitudes. This follows from the simple factorization properties of the generating function Ω_n and the differential operators in (3.15). To exhibit these properties we split the set of $8n$ η_{iA} into two subsets, namely a subset η_{ia} in which A is restricted to index values $A \to a = 1, 2, 3, 4$ and a subset η_{ir} in which $A \to r = 5, 6, 7, 8$. Remarkably, and very simply, the supergravity generating function $\Omega_n(\eta_{iA})$ factors as

$$\Omega_n \;=\; \frac{M_n(1^-, 2^-, 3^+, \ldots, n^+)}{\langle 1\, 2\rangle^8}\, \delta^{(8)}\Big(\sum_{i=1}^{n} |i\rangle\, \eta_{ia}\Big)\, \delta^{(8)}\Big(\sum_{j=1}^{n} |j\rangle\, \eta_{jr}\Big)\,. \tag{3.18}$$

Remarkably and equally simply, the differential operators factorize precisely in accordance with the map (1), including all signs. As an example, we write the map of graviphoton operators with mixed SU(4) indices to illustrate how the $-$ sign in the negative helicity sector arises:

$$b_+^{ar}(i) \leftrightarrow \frac{\partial^2}{\partial\eta_{ia}\partial\eta_{ir}} \;=\; \frac{\partial}{\partial\eta_{ia}}\,\frac{\partial}{\partial\eta_{ir}} \;\;\leftrightarrow\;\; F_+^a(i)\,\tilde{F}_+^r(i)\,, \tag{3.19}$$

$$\begin{aligned} b^-_{ar}(i) &\leftrightarrow \frac{1}{6!}\binom{6}{3}\,\epsilon_{arbcdstu}\,\frac{\partial^6}{\partial\eta_{ib}\partial\eta_{ic}\partial\eta_{id}\partial\eta_{is}\partial\eta_{it}\partial\eta_{iu}} \\ &\leftrightarrow -\Big(-\frac{1}{3!}\epsilon_{abcd}\frac{\partial^3}{\partial\eta_{ib}\partial\eta_{ic}\partial\eta_{id}}\Big)\Big(-\frac{1}{3!}\epsilon_{rstu}\frac{\partial^3}{\partial\eta_{is}\partial\eta_{it}\partial\eta_{iu}}\Big) \;\;\leftrightarrow\;\; -\,F_a^-(i)\,\tilde{F}_r^-(i)\,. \end{aligned} \tag{3.20}$$

We have checked that all $\mathcal{N} = 8$ supergravity operators factor correctly. This also implies that the differential operators (3.16) satisfy the $\mathcal{N} = 8$ supersymmetry algebra.

The factorized structure ensures many desiderata, namely

a. Supergravity amplitudes satisfy $\mathcal{N} = 8$ supersymmetry Ward identities, and they are SU(8) covariant.

b. The spin dependence of $\mathcal{N} = 8$ supergravity amplitudes for all helicity states factorizes into products of gauge theory spin factors. This works for MHV amplitudes because the spin factors obtained by applying differential operators to the product of $\delta^{(8)}$-functions in (3.18) are the *same* for all permutations in formulas such as the KLT formula or the formula (3.21) below, which relate the graviton amplitude M_n to products of two n-gluon amplitudes A_n.

c. $\mathcal{N} = 8$ supersymmetry and SU(8) global symmetry can indeed be implemented in the map (1.1).

These statements have been checked in a number of examples. We discuss some in the next section.

3.4 Tests of the operator map

We now discuss the construction of two examples of MHV amplitudes in $\mathcal{N} = 8$ supergravity from the map (1.1) using the operator correspondence in table 1. We need an explicit formula which relates the n-graviton amplitude to products of n-gluon amplitudes. The KLT formula is available and will be used in the NMHV sector. However, for MHV amplitudes, there is a simpler choice, namely the form recently derived [25] by rearrangement of the BGK formula [24, 8]. It reads

$$M_n(1^-, 2^-, 3^+, \ldots, n^+) = \sum_{\mathcal{P}(i_4,\ldots,i_n)} \frac{\langle 1\,2\rangle\langle i_3\, i_4\rangle}{\langle 1\, i_3\rangle\langle 2\, i_4\rangle}\, s_{1i_n} \left(\prod_{s=4}^{n-1} \beta_s\right) A_n(1^-, 2^-, i_3^+, \ldots, i_n^+)^2\,, \tag{3.21}$$

$$\beta_s = -\frac{\langle i_s\, i_{s+1}\rangle}{\langle 2\, i_{s+1}\rangle}\, \langle 2|\, i_3 + i_4 + \cdots + i_{s-1} |i_s]\,. \tag{3.22}$$

To apply (3.21), one chooses one distinguished positive helicity line i_3 and then sums over permutations of the remaining $n-3$ such lines. This formula embodies the identifications $b^- \leftrightarrow B^- \otimes \tilde{B}^-$ and $b_+ \leftrightarrow B_+ \otimes \tilde{B}_+$ in the operator map of of table 1.

As the first example, we consider the two gravitino MHV amplitude $\big\langle b^-(1)\, f_A^-(2)\, b_+(3) \ldots f_+^B(k) \ldots b_+(n)\big\rangle$ which was obtained in the first line of (2.36) by solving the relevant $\mathcal{N} = 8$ SUSY Ward identity. For a non-zero result, the SU(8) indices must be chosen in the same SU(4) factor, $A, B \to a, b$. For each permutation in (3.21) we make use table 1 to decompose $f_a^- \leftrightarrow F_a^- \otimes \tilde{B}^-$ and write

$$\begin{aligned}
&\big\langle B^-\, F_a^-(2)\, B_+(3) \ldots F_+^b(k) \ldots B_+(n)\big\rangle_L \big\langle \tilde{B}^-\, \tilde{B}^-(2)\, \tilde{B}_+(3) \ldots \tilde{B}_+(k) \ldots \tilde{B}_+(n)\big\rangle_R \\
&\quad = \delta_a^b\, \frac{\langle 1\, k\rangle}{\langle 1\, 2\rangle} \langle B^-(1) B^-(2) B_+(3) \ldots B_+(n)\rangle_L \big\langle \tilde{B}^-\, \tilde{B}^-(2)\, \tilde{B}_+(3) \ldots \tilde{B}_+(k) \ldots \tilde{B}_+(n)\big\rangle_R\,.
\end{aligned}$$

The spin factor can be obtained either from the $\mathcal{N} = 4$ Ward identity, see (2.30), or from the gauge theory generating function. The spin factor $\langle 1\, k\rangle/\langle 1\, 2\rangle$ is common to all permutations in (3.21) and may be extracted as an overall factor. The result via the map (1.1) therefore agrees with the supergravity formula in (2.36).

The next example is the two scalar MHV amplitude given in the fourth line of (2.36). There are three distinct decompositions of the SU(8) indices into distinct SU(4) sectors, and we consider each in turn. It is interesting to note how products of rather different gauge theory amplitudes conspire to produce the common spin factor required by supergravity.

Choose first all group indices in one SU(4), say $\mathrm{SU}(4)_L$, so that $b_{abcd} \leftrightarrow B^- \otimes \tilde{B}_+$. Then (with momentum labels implicit by order) we have

$$\begin{aligned}\big\langle b^-\, b^-_{abcd}\, b_+^{efgh}\, b_+ \ldots b_+\big\rangle &\to \alpha_4^2\, \epsilon_{abcd}\, \epsilon^{efgh} \big\langle B^-\, B^+\, B^-\, B_+ \ldots B_+\big\rangle_L \big\langle \tilde{B}^-\, \tilde{B}^-\, \tilde{B}_+ \tilde{B}_+ \ldots \tilde{B}_+\big\rangle_R \\ &= 4!\, \delta^{efgh}_{abcd}\, \frac{\langle 13\rangle^4}{\langle 12\rangle^4} \big\langle B^-\, B^-\, B_+\, B_+ \ldots B_+\big\rangle_L \big\langle \tilde{B}^-\, \tilde{B}^- \tilde{B}_+ \tilde{B}_+ \ldots \tilde{B}_+\big\rangle_R .\end{aligned}$$

In the second line we have used the gluon MHV identity (2.33) to obtain the spin factor $\langle 1\,3\rangle^4/\langle 1\,2\rangle^4$ (which is common to all permutations in the formula (3.21)). The identity $\epsilon_{abcd}\epsilon^{efgh} = 4!\, \delta^{efgh}_{abcd}$ is also used. The result agrees perfectly with (2.36).

Next split the SU(8) indices such that one leg of each scalar lies in the $\mathrm{SU}(4)_R$ and the rest in $\mathrm{SU}(4)_L$. Reducing the 4-index antisymmetrizer in (2.36) this way gives $\delta^{efgs}_{abcr} = 3!\, \delta^s_r\, \delta^{efg}_{abc}$. The operator map in table 1 tells us

$$\begin{aligned}&\big\langle b^-\, b_{abcr}\, b^{efgs}\, b_+ \ldots b_+\big\rangle \\ &\quad \to (-1)\, \alpha_4^2\, \epsilon_{abcd}\, \epsilon^{efgh} \big\langle B^-\, F_+^d\, F_h^-\, B_+ \ldots B_+\big\rangle_L \big\langle \tilde{B}^-\, \tilde{F}_r^-\, \tilde{F}_+^s\, \tilde{B}_+ \ldots \tilde{B}_+\big\rangle_R \\ &\quad = \; -\, \epsilon_{abcd}\, \epsilon^{efgh} \bigg(-\, \delta^h_d\, \frac{\langle 12\rangle}{\langle 13\rangle}\bigg) \big\langle B^-\, B_+\, B^-\, B_+ \ldots B_+\big\rangle_L \; \times \\ &\qquad\qquad \times \bigg(\delta^s_r\, \frac{\langle 13\rangle}{\langle 12\rangle}\bigg) \big\langle \tilde{B}^-\, \tilde{B}^-\, \tilde{B}_+\, \tilde{B}_+ \ldots \tilde{B}_+\big\rangle_R \\ &\quad = \; 3!\, \delta^s_r\, \delta^{efg}_{abc}\, \frac{\langle 13\rangle^4}{\langle 12\rangle^4} \big\langle B^-\, B^-\, B_+\, B_+ \ldots B_+\big\rangle_L \big\langle \tilde{B}^-\, \tilde{B}^-\, \tilde{B}_+ \tilde{B}_+ \ldots \tilde{B}_+\big\rangle_R .\end{aligned}$$

The minus sign (-1) in the first line comes from conscientiously moving $\tilde{F}_r^-$ past F_h^- when separating the operators into the L and R gauge theory amplitudes. In the second line we used the gluino Ward identity (2.30). In the last line, (2.33) again gives the correct overall spin factor. Observe how gauge theory results (either from Ward identities or the generating function) combine to produce the supergravity amplitude which agrees with (2.36).

The third distinct split of the scalar SU(8) indices places two of the four indices in $\mathrm{SU}(4)_L$ and the other two in $\mathrm{SU}(4)_R$. The antisymmetrizer splits as $\delta^{cdtu}_{abrs} = (2!)^2\, \delta^{cd}_{ab}\, \delta^{tu}_{rs}$. Table 1 tells us that the gauge theory amplitudes needed in (3.21) involve two scalars and $n-2$ gluons. This amplitude is given in (2.31) and contains the spin factor $\langle 1\,3\rangle^2/\langle 1\,2\rangle^2$. In the product of the two gauge theory amplitudes this factor is squared exactly as needed to agree with (2.36).

Several other examples of MHV amplitudes in supergravity have been studied using the map in table 1 to identify the appropriate gauge theory amplitudes. In every case the application of (3.21) produces the same result as a straightforward calculation using the supergravity generating function.

4. An application: intermediate state helicity sums

So far the generating function has been shown to be a useful bookkeeper for the spin dependence of MHV amplitudes in gauge theory and supergravity. In this section we outline a further application, namely to sums over intermediate helicity states needed when the product of MHV trees occurs in a unitarity cut of a 1-loop amplitude.

First we use the generating function to reproduce the intermediate state sum in a 2-particle cut in gauge theory, as discussed in section 5 of [26]. Figure 1(a)[8] indicates the 2-particle cut of a 1-loop amplitude containing MHV amplitudes to the left and right. Each amplitude contains one negative helicity gluon, on line i^- in the left factor and line j^- on the right, plus arbitrary numbers of positive helicity gluons denoted as lines m on the left and n on the right. The intermediate state is a pair of particles of momenta l_1, l_2. Conservation laws allow these to be either a gluon pair, a gluino pair, or a pair of scalars. In the approach of [26], one must solve the Ward identities to find the amplitudes and sum their contributions weighted by the multiplicities, 1-4-6-4-1, of the states in the $\mathcal{N} = 4$ gauge theory. This is not difficult, nor is the resulting binomial sum and spinor algebra which is required to obtain the final answer. However, we find it interesting to put the generating function to work on the problem.

We are interested only in the helicity sum so we drop the dynamical prefactors in the generating function (3.1). The core situation is then governed by the product

$$D_1 D_2\, \delta^{(8)}(I)\delta^{(8)}(J)\,, \tag{4.1}$$

where

$$I = |l_1\rangle\eta_{1a} + |-l_2\rangle\eta_{2a} + |i\rangle\eta_{ia} + \sum_m |m\rangle\eta_{ma}\,, \tag{4.2}$$

$$J = |-l_1\rangle\eta_{1a} + |l_2\rangle\eta_{2a} + |j\rangle\eta_{ja} + \sum_n |n\rangle\eta_{na}\,, \tag{4.3}$$

$$D_l = \prod_{a=1}^{4} \frac{\partial}{\partial\eta_{la}}, \qquad l = 1, 2\,. \tag{4.4}$$

We see that I and J are the arguments of the δ-functions in the generating functions for the left and right amplitudes respectively. The derivatives $D_1 D_2$ act on the Grassmann variables η_{1a} and η_{2a} in both I and J. They reproduce the intermediate state sum in a very compact fashion, automatically keeping track of phases and multiplicities. Each intermediate state comes from a particular split of the individual derivatives in $D_1 D_2$ so that some factors act on I and the rest on J.

To see this first note that, because of the outgoing line convention, the particles on the two ends of an internal line have opposite helicity. One term of the spin sum is the case where a positive helicity gluino F_+^b, with SU(4) index b, is emitted from the left on line l_1 and absorbed as a negative helicity gluino F_b^- on the right. This case corresponds to the split of the operator D_1 with $\partial/\partial\eta_{1b}$ acting on I and the third order D_{1b} from

[8] This reproduces figure 4 of [26].

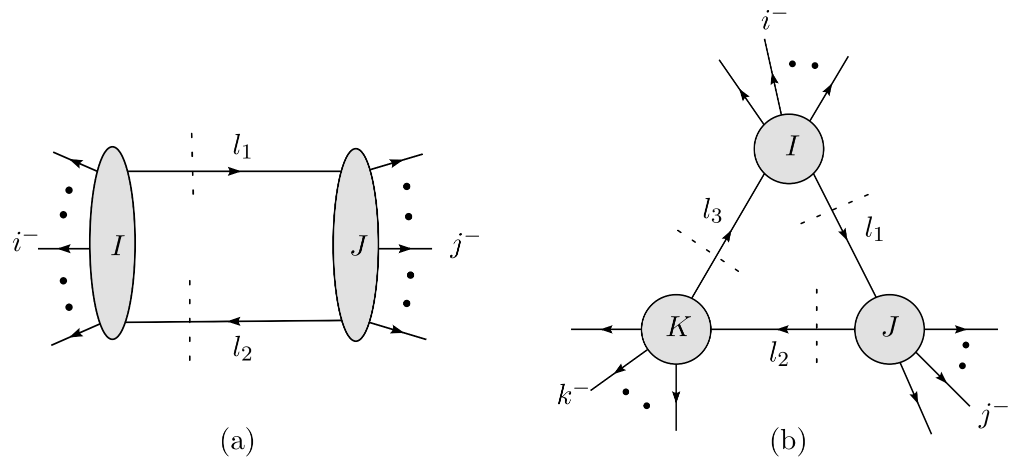

Figure 1: Intermediate spin sums.

the list in (3.3) acting on J. After the 4th order D_i is applied to describe the emission of the negative helicity external gluon, we must apply 3 further derivatives to $\delta^{(8)}(I)$ to have a non-vanishing result. Thus the derivative D_2 is forced to split with the third order D_{2b} applied to I and the first order D_2^b applied to J. The negative sign associated with the fermion loop comes from anti-commutation of derivatives. The multiplicity factor 4 for gluinos comes from the sum over the 4 choices of the index b. This description is unnecessarily tedious. In practice all of the bookkeeping is done automatically (while the physicist sips his tea).

Let's now proceed to the full calculation; we must compute

$$D_1 D_2 \big(D_i \delta^{(8)}(I)\, D_j \delta^{(8)}(J) \big)\,. \tag{4.5}$$

The computation is simpler in the order indicated. We write[9]

$$\begin{aligned} D_i \delta^{(8)}(I) &= D_i \prod_{a=1}^{4} \big(-\langle l_1\, l_2 \rangle \eta_{1a}\eta_{2a} + \langle l_1\, i\rangle \eta_{1a}\eta_{ia} - \langle l_2\, i\rangle \eta_{2a}\eta_{ia} + \dots \big) \\ &= \prod_{a=1}^{4} \big(-\langle i\, l_1\rangle \eta_{1a} + \langle i\, l_2\rangle \eta_{2a} + \dots \big). \end{aligned} \tag{4.6}$$

The omitted terms ... involve the Grassmann variables η_{ma}. They can be dropped since no derivatives $\partial/\partial\eta_{ma}$ will be applied. Hence

$$\begin{aligned} D_i \delta^{(8)}(I)\, D_j \delta^{(8)}(J) &= \prod_{a=1}^{4} \prod_{b=1}^{4} X_a Y_b = (X_1Y_1)(X_2Y_2)(X_3Y_3)(X_4Y_4)\,, \\ X_a &= -\langle i\, l_1\rangle \eta_{1a} + \langle i\, l_2\rangle \eta_{2a}\,, \\ Y_a &= \langle j\, l_1\rangle \eta_{1a} - \langle j\, l_2\rangle \eta_{2a}\,. \end{aligned} \tag{4.7}$$

[9]We use the continuation $|-l\rangle = -|l\rangle$ of spinors for negative null momenta.

Each product simplifies by the Schouten identity, viz

$$\begin{aligned} X_a Y_a &= \big(\langle j\, l_1\rangle\langle i\, l_2\rangle \;-\; \langle j\, l_2\rangle\langle i\, l_1\rangle\big)\eta_{1a}\eta_{2a} \qquad \text{(no sum)} \\ &= -\langle i\, j\rangle\langle l_1\, l_2\rangle\, \eta_{1a}\eta_{2a}\,. \end{aligned} \tag{4.8}$$

Finally we obtain

$$D_1 D_2\big(D_i\delta^{(8)}(I)\, D_j\delta^{(8)}(J)\big) \;=\; \langle i\, j\rangle^4\langle l_1\, l_2\rangle^4, \tag{4.9}$$

which is equivalent to (5.6) of [26]. We did this calculation in gauge theory to facilitate comparison with [26], but it is just as easy in supergravity. The final result there is $\langle i\, j\rangle^8\langle l_1\, l_2\rangle^8$.

It is no more difficult to handle the spin sum for the 3-particle cut shown in figure 1(b), which is related to the supergravity calculation discussed in section 4B of [8]. The external states involve one negative helicity graviton on each sub-amplitude. The core involves a product of three generating δ-functions to which operators $D_1D_2D_3$ which effect the automatic spin sum are applied:

$$D_1D_2D_3\big[\delta^{(16)}(I)\delta^{(16)}(J)\delta^{(16)}(K)\big]\,, \tag{4.10}$$

$$\begin{aligned} I &= |l_1\rangle\eta_{1a} + |-l_3\rangle\eta_{3a} + |i\rangle\eta_{ia} + \sum_m |m\rangle\eta_{ma}\,, \\ J &= |-l_1\rangle\eta_{1a} + |l_2\rangle\eta_{2a} + |j\rangle\eta_{ja} + \sum_n |n\rangle\eta_{na}\,, \\ K &= |-l_2\rangle\eta_{2a} + |l_3\rangle\eta_{3a} + |k\rangle\eta_{ka} + \sum_p |p\rangle\eta_{pa}\,. \end{aligned}$$

The differential operators are now all eighth order, given by the last entry in the list (3.15). Derivatives D_i, D_j, D_k for the external gravitons require only simple calculations similar to (4.6) which give

$$\begin{aligned} D_i\delta^{(8)}(I) &= \prod_{a=1}^{8}\big(\langle i\, l_1\rangle\eta_{1a} - \langle i\, l_3\rangle\eta_{3a}\big) \;\equiv\; \prod_{a=1}^{8} X_a\,, \\ D_j\delta^{(8)}(J) &= \prod_{b=1}^{8}\big(-\langle j\, l_1\rangle\eta_{1b} + \langle j\, l_2\rangle\eta_{2b}\big) \;\equiv\; \prod_{b=1}^{8} Y_b\,, \\ D_k\delta^{(8)}(K) &= \prod_{c=1}^{8}\big(-\langle k\, l_2\rangle\eta_{2c} + \langle k\, l_3\rangle\eta_{3c}\big) \;\equiv\; \prod_{c=1}^{8} Z_c\,. \end{aligned} \tag{4.11}$$

Next we assemble the product

$$\prod_{a=1}^{8}(X_aY_aZ_a) = \prod_{a=1}^{8}\big[\langle i\, l_1\rangle\langle j\, l_2\rangle\langle k l_3\rangle - \langle i\, l_3\rangle\langle j\, l_1\rangle\langle k l_2\rangle\big]\eta_{1a}\eta_{2a}\eta_{3a}\,. \tag{4.12}$$

Finally we apply $D_1D_2D_3$ which trivially gives the result

$$\big[\langle i\, l_1\rangle\langle j\, l_2\rangle\langle k l_3\rangle - \langle i\, l_3\rangle\langle j\, l_1\rangle\langle k l_2\rangle\big]^8 \tag{4.13}$$

and agrees[10] with (4.23) of [8].

We have applied the generating function to situations which are fairly straightforward in their original form in [26] and [8]. However, we wanted to strut our stuff in the hope that the technique will be useful in more complex situations where intermediate spin sums are required.[11]

5. Spin factors and CFT correlators

There is a spectacular analogy between the spin factors for MHV diagrams and holomorphic correlators in conformal field theory on the complex plane. Suppose we are interested in the spin factor for a general MHV n-point amplitude in supergravity which we denote by $\langle\phi_1\phi_2\ldots\phi_n\rangle$. Each of the operators ϕ_1 has an "η-count" $r_i \in 0,1,\ldots,8$ and a specific assignment of SU(8) indices which we omit in the notation. Of course $\sum_i r_i = 16$ for an MHV process as we have emphasized in section 3.3, but this constraint will not play a major role. The conformal analogy we now develop is equally valid in the gauge theory.

The spin factor is defined in this section as

$$\langle\phi_1\phi_2\ldots\phi_n\rangle \;=\; \mathcal{D}_1^{(r_1)}\ldots\mathcal{D}_n^{(r_n)}\delta^{(16)}\Big(\sum_i |i\rangle\eta_{\eta_{iA}}\Big)\,, \tag{5.1}$$

in which $\mathcal{D}_i^{(r_i)}$ is a differential operator of order r_i which carries the SU(8) indices of ϕ_i. In every case we deal with below we assume that the 16 SU(8) indices are paired so that the spin factor is non-vanishing. We noticed the analogy by asking the question "What features of the spin factor are determined only by the r_i and what features require the explicit assignment of indices?"

Let's begin with the 3-point case in which the derivatives in (5.1) give the result (up to a sign):

$$\langle\phi_1\,\phi_2\,\phi_3\rangle \;=\; \langle 1\,2\rangle^{\nu_{12}}\langle 2\,3\rangle^{\nu_{23}}\langle 3\,1\rangle^{\nu_{31}}\,. \tag{5.2}$$

Since each derivative $\mathcal{D}_i^{(r_i)}$ in (5.1) produces r_i factors of the spinor $|i\rangle$, we see that

$$\begin{aligned}\nu_{12}\,+\,\nu_{31} &= r_1\,,\\ \nu_{12}\,+\,\nu_{23} &= r_2\,,\\ \nu_{23}\,+\,\nu_{31} &= r_3\,,\end{aligned} \tag{5.3}$$

which uniquely determine the values

$$\nu_{ij} = \frac{1}{2}(r_i + r_j - r_k)\,, \tag{5.4}$$

where i,j,k is a cyclic permutation of 1,2,3. At this point, the reader will undoubtedly recall that the correlation function of 3 conformal primary operators $\mathcal{O}_i$ of scale dimension $(r_i,0)$ is

$$\langle\mathcal{O}_1(z_1)\,\mathcal{O}_2(z_2)\,\mathcal{O}_3(z_3)\rangle \;=\; c_{123}\,\frac{1}{z_{12}^{\nu_{12}}\,z_{23}^{\nu_{23}}\,z_{31}^{\nu_{31}}} \tag{5.5}$$

[10] The denominator in (4.23) is included in the prefactors omitted in our calculation.

[11] Pilot calculations of 1- and 2-loop helicity sums involving NMHV tree amplitudes in $\mathcal{N} = 4$ SYM indicate that the generating function method is applicable.

with *the same exponents* ν_{ij}. Conclusion: a 3-point spin factor is completely determined by the r_i just as a CFT 3-point correlator is completely determined by the 3 scale dimensions. The forms are strikingly similar. This is not an accident; we can push further.

The spin factor of any 4-point amplitude obtained from (5.1) contains a product of as many as 6 angle brackets, viz.

$$\langle \phi_1\, \phi_2\, \phi_3\, \phi_4 \rangle \;=\; \langle 1\,2 \rangle^{\nu_{12}} \langle 1\,3 \rangle^{\nu_{13}} \langle 1\,4 \rangle^{\nu_{14}} \langle 2\,3 \rangle^{\nu_{23}} \langle 2\,4 \rangle^{\nu_{24}}\, \langle 3\,4 \rangle^{\nu_{34}}\,. \tag{5.6}$$

The set of 4 equations analogous to (5.3) are not sufficient to solve for the 6 exponents ν_{ij}. What else can we do to help determine them? Consider the spin factor for the MHV amplitude corresponding to the partition $r_1 = 7$, $r_2 = 5$, $r_3 = 2$, $r_4 = 2$, which corresponds to an amplitude with one gravitino, one graviphotino, and two graviphotons. Let's write one possible expression which carries the correct scaling weight $(|i\rangle)^{r_i}$ for each spinor, namely

$$\langle \phi_1\, \phi_2\, \phi_3\, \phi_4 \rangle \;\sim\; \langle 1\,2 \rangle^5 \langle 1\,3 \rangle \langle 1\,4 \rangle \langle 3\,4 \rangle\,. \tag{5.7}$$

There is additional freedom to multiply this by a function which is *invariant* under scaling of all 4 spinors. It seems that we can multiply by any function of the variables

$$\xi \;=\; \frac{\langle 1\,3 \rangle \langle 2\,4 \rangle}{\langle 1\,2 \rangle \langle 3\,4 \rangle}\,, \qquad \xi' \;=\; \frac{\langle 2\,3 \rangle \langle 4\,1 \rangle}{\langle 1\,2 \rangle \langle 3\,4 \rangle}\,, \tag{5.8}$$

but they are not independent, rather $\xi' = 1 - \xi$ due to the Schouten identity. Similarly $\xi" = \langle 1\,3 \rangle \langle 2\,4 \rangle / \langle 2\,3 \rangle \langle 4\,1 \rangle = \xi/(1-\xi)$. Thus it appears that the most general form for our spin factor is

$$\langle \phi_1\, \phi_2\, \phi_3\, \phi_4 \rangle \;=\; \langle 1\,2 \rangle^5 \langle 1\,3 \rangle \langle 1\,4 \rangle \langle 3\,4 \rangle\, f(\xi)\,, \tag{5.9}$$

where $f(\xi)$ is an arbitrary function of ξ. At this point the relevance of conformal field theory is clear. The properties of the 4-point spin factor are identical to those of the 4-point correlator[12] of operators with scale dimension $(r_i, 0)$ which involves an arbitrary function of one "cross ratio" which may be taken to be $\Xi = (z_{13})(z_{24})/(z_{12})(z_{34})$.

One property of spin factors, which is not present in conformal field theory, is that the exponents ν_{ij} in (5.6) must be non-negative integers. This severely restricts the choice of $f(\xi)$ to $f(\xi) = 1$, $f(\xi) = \xi$, or $f(\xi) = 1 - \xi$. Each choice corresponds to an inequivalent configuration of SU(8) labels as follows:

$$\begin{aligned} f(\xi) = 1 \quad &\leftrightarrow \quad \langle \phi_1^{1234567}\, \phi_2^{12345}\, \phi_3^{68}\, \phi_4^{78} \rangle\,, \\ f(\xi) = \xi \quad &\leftrightarrow \quad \langle \phi_1^{1234567}\, \phi_2^{12348}\, \phi_3^{56}\, \phi_4^{78} \rangle\,, \\ f(\xi) = 1 - \xi \quad &\leftrightarrow \quad \langle \phi_1^{1234567}\, \phi_2^{12348}\, \phi_3^{68}\, \phi_4^{57} \rangle\,. \end{aligned} \tag{5.10}$$

A general n-point spin factor for a process involving operators of η-count $r_1, r_2, \ldots r_n$ can be expressed as the product of up to $n(n-1)/2$ independent angle brackets $\langle i\, j \rangle$, each raised to the non-negtive integer power ν_{ij}. Suppose that we have obtained one candidate

[12] See for example section 5.1 of [27]. The scale dependent product of six factors $(\langle i\, j \rangle)^{r/3 - r_i - r_j}$ suggested by (5.28) of [27] can be used for the general 4-point spin factor. However, it involves fractional exponents, since $r = \sum_i r_i \;=\; 16$, which is awkward.

expression which scales as Λ^{r_i} for each spinor. We would then need to consider modification of that expression, involving the possible scale invariant variables which can be constructed from the spinors. To find such variables it is sufficient to scale each spinor[13] to the form $|i\rangle \to \tilde{\lambda}_i^{\dot{\alpha}} = (z_i \ 1)$. Then each angle bracket satisfies $\langle i\, j\rangle = z_i - z_j \equiv z_{ij}$ This establishes an exact correspondence between scale invariant variables constructed from spinor angle bracket and CFT cross ratios. In the 4-point case above, we have $\xi = \Xi$! There are $n-3$ independent variables for an n-point function.

Although the CFT analogy is quite perfect, it has been of limited use for us. One application concerns the asymptotic behavior of the spin factors for diagrams in the MHV-vertex expansion of an NMHV amplitude.

6. Generating Functions for NMHV amplitudes

We would like to extend our study beyond the MHV sector, but there are several difficulties. The structure of non-MHV amplitudes with external gluons or gravitons is far more complicated than MHV, and the recursion relations they satisfy contain more terms. It is also more difficult to extract information[14] from the SUSY Ward identities which relate amplitudes within each non-MHV sector. Happily it turns out that we can make considerable progress in the NMHV (next-to-MHV) sector which consists of all amplitudes connected by supersymmetry to the n-gluon or n-graviton amplitude with 3 negative helicity lines. One needs $n \geq 6$ for a genuine NMHV amplitude. For $n = 5$, the amplitude with helicity configuration $\langle - - - + + \rangle$ is the complex conjugate of the MHV configuration $\langle + + + - - \rangle$.

In this section we discuss NMHV amplitudes in $\mathcal{N} = 4$ gauge theory and $\mathcal{N} = 8$ supergravity. Our treatment is based on the MHV-vertex expansion developed for gauge theory in [16] and extended to gravity in [17]. For external gluon amplitudes, the method was established before the invention of modern recursion relations in [9, 10], but the version of recursion relations studied in [18] provides the simplest and most general approach, and clarifies the validity of the method.

6.1 Recursion relations and the MHV-vertex method

Recursion relations express n-point tree amplitudes such as A_n as finite sums of products of two sub-amplitudes A_{n_1}, A_{n_2} with $n_1, n_2 < n$. They exploit the simple analyticity properties of on-shell tree amplitudes in a variable z which appears through a shift of the spinors used to parametrize the complex momenta. Cauchy's theorem can be used to derive a valid recursion relation provided that the amplitude vanishes as the complex variable $z \to \infty$. Our applications to $\mathcal{N} = 8$ supergravity force us to confront this basic fact head on, so it will play an important role in our discussion below (sections 6.2.2 and 6.3.2). Later we will compare the large z behavior associated with both 3-line and the more common 2-line shift, so we begin with a review of the latter. See [30, 8] and [31] for more information on the large z asymptotics.

[13] We thank Gary Gibbons for this observation.

[14] We solve the $\mathcal{N} = 1$ SUSY Ward identities for NMHV 6-point amplitudes in appendix B. Some explicit results have also been found in [28] and [29].

6.1.1 2-line shifts

The simplest recursion relations are based on a complex continuation of on-shell amplitudes in which the spinors of two external lines are shifted. Suppose that we are interested in n-point amplitudes of gluons $A_n(1^-, 2^-, 3, 4, \ldots, n)$ or gravitons $M_n(1^-, 2^-, 3, 4, \ldots, n)$. Particles 1 and 2 have negative helicity, as indicated, while the helicity of the remaining particles can be positive or negative. In the method of [11], the spinors of particles 1 and 2 are shifted as follows:[15]

$$\begin{aligned} |1\rangle \to |\hat{1}\rangle &= |1\rangle - z|2\rangle\,, & |2\rangle &\to |2\rangle\,, \\ |1] \to |1]\,, & & |2] &\to |\hat{2}] = |2] + z|1]\,. \end{aligned} \tag{6.1}$$

The shifted momenta are rank 1 products of spinors and therefore null vectors, and the shift cancels in the sum, so that momentum is conserved, viz

$$(\hat{p}_1 + \hat{p}_2)^{\dot{\alpha}\beta} = \big(|1\rangle - z|2\rangle\big)[1| + |2\rangle\big([2| + z[1|\big) = (p_1 + p_2)^{\dot{\alpha}\beta}\,. \tag{6.2}$$

Therefore the shifted amplitudes

$$A_n(z) = A_n(\hat{1}^-, \hat{2}^-, 3, \ldots, n)\,, \qquad M_n(z) = M_n(\hat{1}^-, \hat{2}^-, 3, \ldots, n)\,. \tag{6.3}$$

are indeed on-shell analytic continuations of $A_n(0) = A_n(1^-, 2^-, 3, \ldots, n)$ and $M_n(0) = M_n(1^-, 2^-, 3, \ldots, n)$.

The only singularities of tree amplitudes are poles where propagators vanish. Therefore $A(z)$ and $M(z)$ are meromorphic with simple poles in z, a pole for each partition of the amplitude into a product of two sub-amplitudes connected by a propagator carrying the z-dependent momentum $\hat{P}_I = \hat{p}_1 + K_1 = -(\hat{p}_2 + K_2)$ where $K_1 = \sum p_{I_1}$ and $K_2 = \sum p_{I_2}$ are the sums of unshifted momenta in each factor. If the important condition that $A(z) \to 0$ as $z \to \infty$ is satisfied, then Cauchy's theorem can be applied to derive, see [10, 11], the recursion relation

$$A_n(1^-, 2^-, 3, \ldots, n) = \sum_I A_{n_1}(\hat{1}^-, -\hat{P}_I, \ldots, n)\frac{1}{s_I}A_{n_2}(\hat{P}_I, \hat{2}^-, 3, \ldots)\,. \tag{6.4}$$

Here $n_1 + n_2 = n + 2$, and $s_I = -(p_1 + K_1)^2$ is the unshifted Mandelstam invariant associated with the partition I. The sum includes all partitions of the amplitude in which the shifted lines are on opposite sides. For each such partition, z is evaluated at the pole z_I determined by

$$0 = \hat{P}_I^2 = (\hat{p}_1 + K_1)^2 = -s_I + z_I\langle 2|P_I|1]\,, \tag{6.5}$$

in which $P_I = p_1 + K_1$. The internal particle emitted from the first sub-amplitude can have either helicity. Propagation to the second sub-amplitude conserves helicity, but it is recorded there with opposite helicity because of the outgoing momentum convention. Graviton amplitudes satisfy recursion relations [11] of the same form as (6.4), but with A_n's replaced by M_n's.

[15] This is usually called a $[-,-\rangle$ shift. It is known that $[-,+\rangle$ and $[+,+\rangle$ shifts also lead to a valid recursion relation, but a $[+,-\rangle$ shift does not.

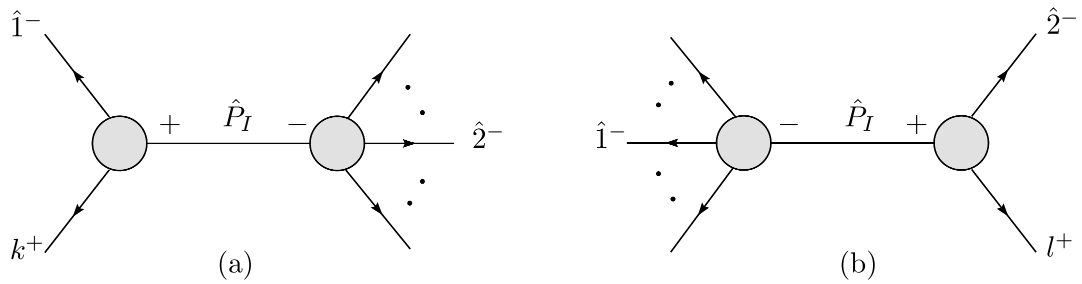

Figure 2: Diagrams from the 2-line shift recursion relations (6.4) for MHV amplitudes in both gauge theory and gravity. The 3-vertex in the right hand diagram vanishes as a consequence of kinematics.

It is interesting to examine the types of diagrams that actually contribute to the recursion relation for various types of amplitudes. The simplest case is MHV amplitudes in gauge theory in which color ordering and helicity conservation (see the discussion below (2.24)–(2.25)) imply that only the two diagrams listed in figure 2 can contribute. Both of them involve the 3-gluon vertex with two positive helicity lines. But the two situations are rather different. The 3-vertex in figure 2(b) vanishes at the pole by "special kinematics": the null condition $\hat{P}_I^2 = \langle 2\,l\rangle[\hat{2}\,l] = 0$ for the internal line requires $[\hat{2}\,l] = 0$, and therefore $A_3(\hat{2}^-, \hat{P}_I^+, l^+) \propto [\hat{P}_I\,l]^3 \propto [\hat{2}\,l]^3 = 0$. The 3-vertex in figure 2(a) does not vanish because $|\hat{1}] = |1]$. Thus there is only one contributing diagram, and it is not difficult to show by iteration [32] that the Parke-Taylor amplitude (3.4) is the solution of the recursion relation. The diagrams for n-graviton MHV amplitudes are similar. Helicity conservation restricts the possible diagrams to those containing a 3-graviton vertex, and special kinematics again forces the diagram of figure 2(b) to vanish. However there are now more diagrams of the type in figure 2(a), namely the $n-2$ diagrams containing cyclic permutations of the positive helicity lines [11]. This is required by Bose symmetry. The simplicity of MHV recursion relations was exploited in [25] to prove a relationship between M_n and $(A_n)^2$ for all n.

The recursion relation is also valid for non-MHV amplitudes, but more diagrams contribute. Diagrams for the simplest case of the NMHV gauge theory amplitudes, such as $A_n(1^-, 2^-, 3^-, 4^+, \ldots, n^+)$, include both NMHV and MHV sub-amplitudes. An example with NMHV vertices is shown in figure 3. This undesirable feature can be avoided with 3-line shifts as we discuss in section 6.1.2.

Recursion relations fail if amplitudes shifted as in (6.1) do not vanish as $z \to \infty$. It is by now well established that n-gluon amplitudes vanish as $1/z$ and n-graviton amplitudes vanish as $1/z^2$ if two negative helicity lines are shifted [30, 8, 31]. Indeed this behavior of MHV amplitudes can be directly observed in (3.4) and (3.21). But the asymptotic behavior does depend on particle type, a fact of particular concern for this paper. For example consider the amplitude (2.31) for two scalars and $n-2$ gluons. If lines 1 and 2 are shifted, the spin factor in (2.31) contains a factor z^2 which overwhelms the $1/z$ falloff of the n-gluon amplitude. One can see from (2.36) that amplitudes in which a pair of gravitons of opposite helicity is replaced by a pair of particles of spin s behave as $z^{(2-2s)}$ at large z. These remarks apply specifically to the shift of (6.1), and there are other shifts available in these examples. In particular, if the spinors of particle 1 and 2 are exchanged in (6.1),

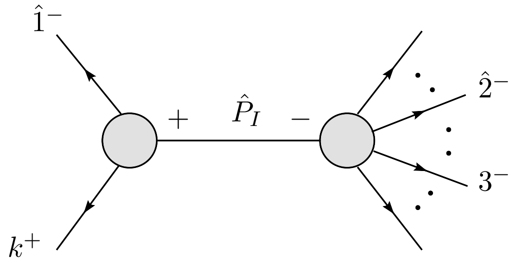

Figure 3: 2-line shift recursion relations for NMHV amplitudes contain non-vanishing diagrams with NMHV vertices. It is an appealing feature of 3-line shifts that such diagrams vanish due to special kinematics and the recursion sum consequently contains MHV subdiagrams only.

the amplitudes vanish as $z \to \infty$ and recursion relations can be derived.

6.1.2 3-line shifts

We now discuss the recursion relation which arises from the 3-line shift first considered in [18] with further details discussed in [17]. This shift applies to NMHV amplitudes such as $A_n(1^-, 2^-, 3^-, 4^+, \ldots, n^+)$ and $M_n(1^-, 2^-, 3^-, 4^+, \ldots, n^+)$. In this shift, all $|p\rangle$ spinors are unchanged, while three spinors $|1]$, $|2]$ $|3]$ are shifted. In our applications we will consider the 123-shift and other choices for the 3 shifted lines. So we define the more general shift

$$\begin{aligned}
|m_1] &\to |\hat{m}_1] = |m_1] + z\,\langle m_2\, m_3\rangle\, |X]\,, \\
|m_2] &\to |\hat{m}_2] = |m_2] + z\,\langle m_3\, m_1\rangle\, |X]\,, \\
|m_3] &\to |\hat{m}_3] = |m_3] + z\,\langle m_1\, m_2\rangle\, |X]\,,
\end{aligned} \tag{6.6}$$

where $|X]$ is an arbitrary reference spinor, which will play an important role in our analysis. Shifted momenta $\hat{p}_i$ remain on shell and total momentum is conserved because of the Schouten identity.

Let's focus first on 6-point amplitudes. When the m_i are chosen to be the three negative helicity lines, the pure gluon and pure graviton amplitudes vanish for large z. Using for example the results for the amplitudes given in the literature, e.g. [28], we find numerically for large z in gauge theory

$$\begin{aligned}
\langle \hat{-}\hat{-}\hat{-}+++\rangle &\sim \langle \hat{-}\hat{-}+\hat{-}++\rangle \;\sim\; \frac{1}{z^4}\,, \\
\langle \hat{-}+\hat{-}+\hat{-}+\rangle &\sim \frac{1}{z^5}\,,
\end{aligned} \tag{6.7}$$

and in gravity (via KLT)

$$\langle \hat{-}\hat{-}\hat{-}+++\rangle \sim \frac{1}{z^6}\,. \tag{6.8}$$

Higher n-point graviton amplitudes are discussed in section 6.3.5.

We will also see later that these exponents change for amplitudes with other states of the $\mathcal{N}=4$ or $\mathcal{N}=8$ theories. However, it is clear that the NMHV amplitudes A_6 and M_6 do satisfy recursion relations. We first discuss the gauge theory case in detail and then the modifications necessary for gravity.

Gauge theory. The NMHV amplitude $A_n(1^-,2^-,3^-,4^+,\dots,n^+)$ satisfies the recursion relation

$$A_n(1^-,2^-,3^-,4^+,\dots,n^+) = \sum_I A_{n_1}(\hat{m}_1,-\hat{P}_I,\dots,n)\frac{1}{s_I}A_{n_2}(\hat{P}_I,\hat{m}_2,\hat{m}_3,4,\dots)\,, \qquad (6.9)$$

in which m_1, m_2, m_3 are a cyclic permutations of 1,2,3 and the sum includes all partitions I in which the negative helicity lines are separated and all external lines appear in cyclic order on the right side.[16] The contribution of each individual diagram depends on $|X]$. However, if the amplitude vanishes as $z \to \infty$ for all $|X]$, Cauchy's theorem ensures that the sum of all diagrams is independent of $|X]$.

Spinors for the shifted momenta $\hat{1},\hat{2},\hat{3},\hat{P}_I$ which appear in (6.9) are evaluated at the pole of the variable z in (6.6) for each contributing diagram. The channel momentum $\hat{P}_I$ can always be written to include the negative helicity line $\hat{m}_1$, plus the sum K of the positive helicity lines in the same sub-amplitude, i.e. $\hat{P}_I = \hat{m}_1 + K$. The pole condition, similar to (6.5), is

$$0 \;=\; \hat{P}_I^2 = -s_I - z_I\langle m_2\, m_3\rangle\langle m_1|P_I|X]\,. \qquad (6.10)$$

Scalar products among the shifted (denoted by m_i) and unshifted (denoted by k) spinors are required to evaluate sub-amplitudes. They are given by (see [17])

$$\langle i\,\hat{P}_I\rangle = \omega^{-1}\,\langle i\,|P_I|X]\,, \qquad \omega = [\hat{P}_I\,X]\,, \qquad (6.11)$$

$$[\hat{P}_I\,k] = \frac{\omega\,\langle m_1|P_I|k]}{\langle m_1|P_I|X]}\,, \qquad P_I = m_1 + K\,, \qquad (6.12)$$

$$[\hat{m}_2\,\hat{m}_3] = [m_2\,m_3] + z_I\,\langle m_1|m_2+m_3|X]\,, \qquad (6.13)$$

$$[\hat{m}_1\,k] = [m_1\,k] - z_I\,\langle m_2\,m_3\rangle[k\,X]\,. \qquad (6.14)$$

where z_I is determined by (6.10).

We now return to the recursion sum (6.9). Due to helicity conservation, the two sub-amplitudes must be MHV for partitions in which both $n_1 > 3$ and $n_2 > 3$. The only possibility for non-MHV subdiagrams in the sum (6.9) arises from diagrams like that of figure 3, but now with $1,2,3$ all shifted according to (6.6). As a result of the different shifts, the 3-point anti-MHV amplitude in this expression now vanishes due to kinematics. This is easily seen using $A_3(\hat{m}_1^-,k^+,-\hat{P}_I^+) = [k\,\hat{P}_I]^3/([\hat{P}_I\,\hat{m}_1][\hat{m}_1\,k])$. Equations (6.12) then tell us that $[k\,\hat{P}_I] = -\omega\,\langle m_1|P_I|k]/\langle m_1|P_I|X] = 0$, because $P_I = m_1 + k$, while both factors in the denominator are non-vanishing. We conclude that the sum (6.9) (and its generalizations to other orderings of the $\pm$ve helicity lines) contains only diagrams where each subdiagram is MHV. This is the principal advantage of the 3-line shift. It allows the construction of the relatively difficult NMHV amplitudes from simpler and familiar MHV elements.

Since the sub-amplitudes are MHV, there is only one choice of helicities for the internal line, and hence each diagram in the recursion expansion is uniquely characterized by its pole momentum P_I. Figure 4 shows a typical diagram $\mathcal{A}_{n,I}$ that contributes to the sum

[16]Formula (6.9) applies specifically to the case of consecutive ordering of the lines of each helicity. For other orderings there are similar relations.

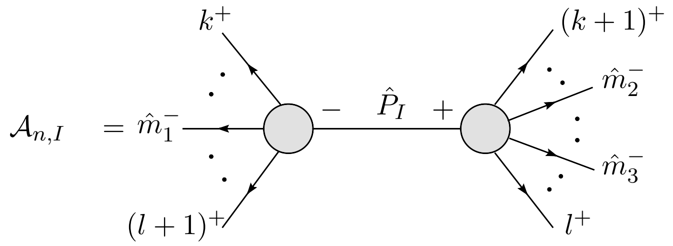

Figure 4: Generic MHV-vertex diagram from the 3-line shift recursion relations for NMHV amplitudes in both gauge theory and gravity.

in (6.9) for $A_n(1,\ldots,m_1^-,\ldots,m_2^-,\ldots,m_3^-,\ldots,n)$. Each vertex amplitude is MHV, so we use the Parke-Taylor formula [23] to write

$$\begin{aligned}\mathcal{A}_{n,I} &= A_{n_1}(\hat{m}_1^-,\ldots,k^+,-\hat{P}_I^-,(l+1)^+,\ldots)\,\frac{1}{s_I}\,A_{n_2}(\hat{m}_2^-,\ldots,\hat{m}_3^-,\ldots,l^+,\hat{P}_I^+,(k+1)^+,\ldots)\\ &= \frac{\langle m_1\,\hat{P}_I\rangle^4}{\langle \hat{P}_I,l+1\rangle\cdots\langle k\,\hat{P}_I\rangle}\,\frac{1}{s_I}\,\frac{\langle m_2\,m_3\rangle^4}{\langle \hat{P}_I,k+1\rangle\cdots\langle l\,\hat{P}_I\rangle}\,. \end{aligned}\tag{6.15}$$

Each angle bracket with $\hat{P}_I$ can be rewritten using (6.11), giving $|\hat{P}_I\rangle = \omega^{-1}P_I\,|X]$. Since the ω-factors cancel in (6.15), we can ignore them from the beginning. Thus we will use the simpler rule

$$|P_I\rangle = P_I|X] = (p_{m_1}+K)|X]\,. \tag{6.16}$$

This is the CSW spinor prescription for an internal line, and the 3-line recursion relations thus reproduce the MHV-vertex expansion of [16].

A useful alternate form [15] of (6.15) is

$$\mathcal{A}_{n,I} \;=\; \left(\prod_1^n\langle i,i+1\rangle^{-1}\right)\frac{1}{V_I}\,\langle m_2\,m_3\rangle^4\,\langle m_1\,P_I\rangle^4\,, \tag{6.17}$$

where

$$\frac{1}{V_I} \;=\; \frac{\langle l,l+1\rangle\langle k,k+1\rangle}{s_I\,\langle P_I,l+1\rangle\langle k\,P_I\rangle\langle P_I,k+1\rangle\langle l\,P_I\rangle}\,. \tag{6.18}$$

The cyclic invariant product is common to all diagrams which contribute to the full amplitude.

Example: the 6-point gluon NMHV amplitudes. Shifting the negative helicity lines $1,2,3$, the recursion relations for the gluon amplitude $A_6(1^-,2^-,3^-,4^+,5^+,6^+)$ contains 6 diagrams which we label by their poles, namely 12, 23, 34, 61, 612, and 234; the diagrams are shown in figure 5. While each diagram depends on $|X]$, their sum is $|X]$-independent because $A_6(z)\to 0$ for all $|X]$ as $z\to\infty$.

The two other 6-point gluon NMHV amplitudes $A_6(1^-,2^-,3^+,4^-,5^+,6^+)$ and $A_6(1^-,2^+,3^-,4^+,5^-,6^+)$ can likewise be computed from recursion relations obtained from shifting the three negative helicity lines. They contain 8 and 9 diagrams, respectively, and again the sums of diagrams are independent of $|X]$.

Let us now consider the analogous approach to graviton amplitudes.

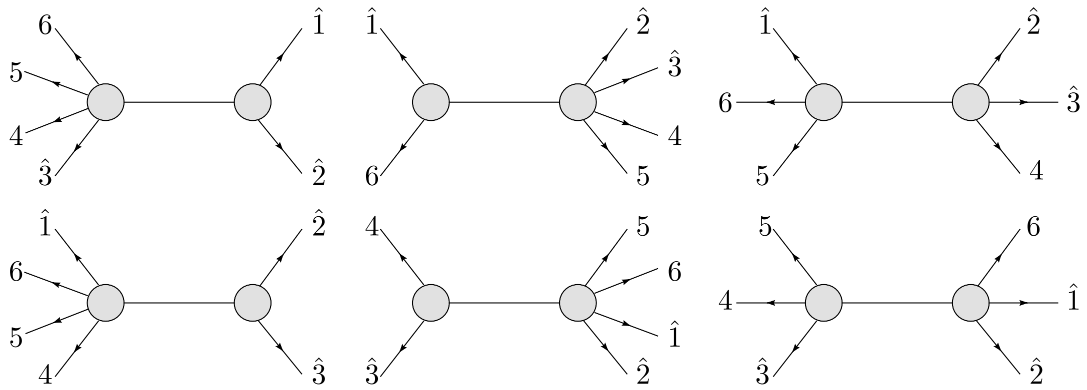

Figure 5: The six MHV-vertex diagrams needed for the 3-line recursion relation for the gluon NMHV amplitude $A_6(1^-, 2^-, 3^-, 4^+, 5^+, 6^+)$. Amplitudes for other external particles of $\mathcal{N} = 4$ theory are obtained by multiplying each gluon diagram by the appropriate spin factor.

Gravity. The 3-line shift gives the recursion relation

$$M_n(1^-, 2^-, 3^-, 4^+, \dots, n^+) = \sum_I M_{n_1}(\hat{m}_1^-, -\hat{P}_I^-, \dots)\, \frac{1}{s_I}\, M_{n_2}(\hat{m}_2^-, \hat{m}_3^-, \hat{P}_I^+, \dots)\,. \tag{6.19}$$

For each value of n_1, the sum includes all cyclic orderings of the negative helicity lines and of all distinct arrangements of the positive helicity lines. Overall Bose symmetry is then maintained. The sum in (6.19) only contains MHV-vertex diagrams. This can be shown as in gauge theory using $M_3 = A_3^2$.

The form of the BGK formula presented in [25] can be used to express the two MHV sub-amplitudes and show that the ω-factor in (6.11) drops out. In more detail:

$$\begin{aligned} &M_{n_1}(\hat{m}_1^-, -\hat{P}_I^-, i_3^+, \dots, i_{n_1}^+) \\ &\quad = \omega^{-4} \sum_{\mathcal{P}(i_4,\dots,i_{n_1})} \frac{\langle m_1\, P_I\rangle \langle i_3\, i_4\rangle}{\langle m_1\, i_3\rangle \langle P_I\, i_4\rangle}\, s_{\hat{m}_1 i_{n_1}} \left(\prod_{s=4}^{n_1-1} \beta_s \right) A_{n_1}(\hat{m}_1^-, -P_I^-, i_3^+, \dots, i_{n_1}^+)^2\,, \end{aligned} \tag{6.20}$$

with

$$\beta_s = -\frac{\langle i_s\, i_{s+1}\rangle}{\langle P_I\, i_{s+1}\rangle}\, \langle P_I|\, i_3 + i_4 + \cdots + i_{s-1} |i_s]\,. \tag{6.21}$$

The ω^{-4} factor comes from setting $|\hat{P}_I\rangle = \omega^{-1} P_I\, |X]$ in $A_{n_1}^2$.

Likewise,

$$\begin{aligned} &M_{n_2}(\hat{m}_2^-, \hat{m}_3^-, \hat{P}_I^+, j_4^+, \dots, j_{n_2}^+) \\ &\quad = \omega^{4} \sum_{\mathcal{P}(j_4,\dots,j_{n_2})} \frac{\langle m_2\, m_3\rangle \langle P_I\, j_4\rangle}{\langle m_2\, P_I\rangle \langle m_3\, j_4\rangle}\, s_{\hat{m}_2 j_{n_2}} \left(\prod_{s=4}^{n_2-1} \beta_s \right) A_{n_2}(\hat{m}_2^-, \hat{m}_3^-, P_I^+, j_4^+, \dots, j_{n_2}^+)^2\,, \end{aligned} \tag{6.22}$$

with

$$\beta_s = -\frac{\langle j_s\, j_{s+1}\rangle}{\langle m_3\, j_{s+1}\rangle}\, \langle m_3|\, \hat{P}_I + j_4 + \cdots + j_{s-1} |j_s]\,. \tag{6.23}$$

This latter expression contains $\hat{P}_I$ only in the ω-independent combination $\langle m_3\, \hat{P}_I\rangle[\hat{P}_I\, j_s] = \langle m_3\, P_I\rangle\langle m_1|P|j_s]/\langle m_1\, P_I\rangle$. (See (6.11)–(6.12).) The two results for ω-factors are only valid for $n_1, n_2 \geq 4$. For n_1 or $n_2 = 3$ one can simply use $M_3 = (A_3)^2$ to deduce the same results. It is obvious now that the ω-factors cancels in $M_{n_1}\, s_I^{-1} M_{n_2}$ yielding diagrams which are independent of ω. Note that the effect of the shift appears in $|P_I\rangle$, given in (6.16), and in $s_{\hat{m}_i j} = \langle m_i\, j\rangle[j\, \hat{m}_i]$.

Example: the 6-point graviton NMHV amplitude. Shifting the negative helicity lines $1, 2, 3$, the recursion relations for the graviton amplitude $M_6(1^-, 2^-, 3^-, 4^+, 5^+, 6^+)$ contains 21 diagrams which fall in three classes: three 2-particle "$--$" poles ($I = 12, 13, 23$), nine 2-particle "$-+$" poles ($I = m_i 4, m_i 5, m_i 6$) and nine 3-particle poles ($I = m_i 45, m_i 46, m_i 56$). One only needs to compute one amplitude from each class; the rest can be obtained by momentum relabelling. Our numerical check shows that the sum of the 21 diagrams is independent of $|X]$.

The diagrammatic expansions associated with the recursion relations (6.9) for gauge theory and (6.19) for gravity are the basis for our treatment of NMHV amplitudes. We apply them to amplitudes for general external states of $\mathcal{N} = 4$ SYM and $\mathcal{N} = 8$ supergravity using the generating functions discussed below to determine the spin factors for each diagram. It is important that the amplitudes vanish as $z \to \infty$, and this condition will play a crucial role in the application to $\mathcal{N} = 8$ supergravity.

6.2 NMHV Generating Function for $\mathcal{N} = 4$ SYM

In this section we derive the generating function of [15] and discuss several properties that are important for our application. The goal is to obtain a correct and efficient construction of the entire NMHV sector of the $\mathcal{N} = 4$ theory. The NMHV sector consists of the top n-gluon amplitudes A_n (for various orderings of the three negative helicity lines) together with all other amplitudes related to those by SUSY Ward identities. The practical definition of this sector is that it contains all sets of external particles for which the corresponding differential operator formed from products of n factors from the list (3.3) is of total order 12, rather than order 8 which characterizes the MHV sector. There is another significant difference between the two sectors. In the MHV sector there is a single generating function F_n, given in (3.1), from which all n-point amplitudes are obtained. In the NMHV sector there is a different generating function for each diagram $\mathcal{A}_{n,I}$ in the MHV-vertex decomposition. After applying the appropriate 12th order differential operator, the full amplitude is obtained by adding the results for all diagrams contributing to the recursion relation (6.9).

Consider an NMHV n-point amplitude for a general set of external states, and choose 3 lines, m_1, m_2, m_3, to shift, as in (6.6), such that the amplitude vanishes as $z \to \infty$. Not all shifts produce an amplitude with the required falloff at large z. The issue of the choice of a valid shift is discussed in section 6.2.2.

Given a valid shift, the amplitude can be expressed as the sum of diagrams in the recursion relation (6.9). Each diagram contains the product of two MHV sub-amplitudes, as shown in figure 6, and each of these can be expressed as the appropriate eighth order product of derivatives from the correspondence (3.3) acting on the MHV generating func-

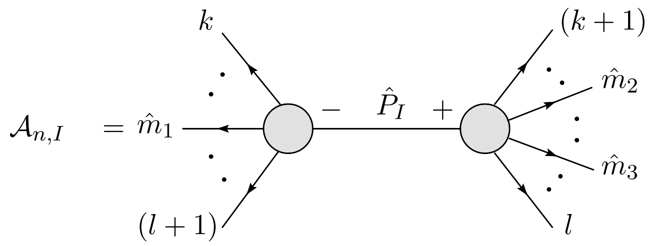

Figure 6: MHV-vertex diagram from the 3-line shift recursion relations for NMHV amplitudes with external states of the $\mathcal{N}=4$ theory.

tion (3.1). Thus we can write the generalization of the amplitude (6.15) of figure 6 to an arbitrary set of external states as

$$\mathcal{A}_{n,I} \equiv \mathrm{sign}(I)\, \frac{A_{n_1}(l+1,\ldots,m_1,\ldots,k,-P_I)}{\langle m_1\, P_I\rangle^4} \frac{1}{s_I} \frac{A_{n_2}(P_{I'},k+1,\ldots,m_2,\ldots,m_3,\ldots,l)}{\langle m_2 m_3\rangle^4} \times \Big(D_{l+1}\ldots D_k\, D_I\, \delta^{(8)}\big(L\big)\Big) \Big(D_{I'} D_{k+1}\ldots D_l\, \delta^{(8)}\big(R\big)\Big)\,, \tag{6.24}$$

where

$$L \;=\; |P_I\rangle\, \eta_{Ia} + \sum_{i=l+1}^{k} |i\rangle\, \eta_{ia}\,, \qquad R \;=\; |P_I'\rangle\, \eta_{Ib} + \sum_{i=k+1}^{l} |i\rangle\, \eta_{ib}\,. \tag{6.25}$$

The delta functions $\delta^{(8)}$ are defined in (3.2). The spinors for the internal lines are[17]

$$|P_I\rangle = P_I|X] \;= -|P_{I'}\rangle\,, \tag{6.26}$$

where P_I is the sum of the external momenta on the left sub-amplitude of figure 6.

The differential operators D_I and $D_{I'}$ represent particles at the left and right ends of the internal line. Since these particles are opposite helicity states of the same field, the orders of the operators are related by $d_I + d_{I'} = 4$, and they carry distinct SU(4) indices. Thus

$$D_I\, D_{I'} = \pm \prod_{a=1}^{4} \frac{\partial}{\partial \eta_{Ia}}\,. \tag{6.27}$$

When the derivative operators of the external lines are applied to $\delta^{(8)}\big(L\big)\delta^{(8)}\big(R\big)$, they uniquely determine the split of the four derivatives $\prod_{a=1}^{4} \frac{\partial}{\partial \eta_{Ia}}$ into the product $D_I\, D_{I'}$. Starting with an initial ordering of the differential operators, $D_1 D_2 \ldots D_n$ as dictated by the color ordering, we can therefore write

$$\left(\prod_{a=1}^{4} \frac{\partial}{\partial \eta_{Ia}}\right) D_1 D_2 \ldots D_n = \mathrm{sign}(I)\, D_{l+1}\ldots D_k\, D_I\, D_{I'}\, D_{k+1}\ldots D_l\,, \tag{6.28}$$

where the sign, $\mathrm{sign}(I) = \pm 1$, which also appeared in (6.24), arises from the required interchange of Grassman derivatives.

We now use three facts:

[17] All factors of ω have been removed as discussed in the previous section.

i. The external state derivatives can all be moved to the left of the expression (6.24) and reordered according to (6.28).

ii. Integration and differentiation are equivalent for functions of Grassmann variables, so the 4 η_{Ia} derivatives can be written as integrals.

iii. The 4-fold integral can be performed using the technique of section 5 of [15].

Using this we can rewrite (6.24) as the product $D_1 D_2 \dots D_n$ of derivatives acting on

$$\int \prod_{a=1}^{4} d\eta_{Ia}\, \delta^{(8)}\big(L\big)\, \delta^{(8)}\big(R\big) \;=\; \delta^{(8)}\bigg(\sum_{i=1}^{n} |i\rangle \eta_{ia}\bigg) \prod_{b=1}^{4} \sum_{j=l+1}^{k} \langle P_I\, j\rangle \eta_{jb}\,. \tag{6.29}$$

We started with a product of 16 derivatives in (6.24) and eliminated the 4 derivatives on η_{Ia}, The remaining product of 12 derivatives corresponds exactly to the external states of the amplitude.

Since the argument applies to any diagram of the general NMHV amplitude, we have derived the generating function

$$\mathcal{F}_{n,I} \;=\; \frac{\mathcal{A}^{\text{gluons}}_{n,I}}{\langle m_1\, P_I\rangle^4 \langle m_2 m_3\rangle^4}\, \delta^{(8)}\bigg(\sum_{i=1}^{n} |i\rangle \eta_{ia}\bigg) \prod_{b=1}^{4} \sum_{j\epsilon I} \langle P_I\, j\rangle \eta_{jb}\,, \tag{6.30}$$

where $\mathcal{A}^{\text{gluons}}_{n,I}$ is the value of the pure gluon MHV-vertex diagram obtained from the same shift,

$$\mathcal{A}^{\text{gluons}}_{n,I} \;=\; A_{n_1}(l+1, \dots, m_1, \dots, k, -P_I)\, \frac{1}{s_I} A_{n_2}(P_{I'}, k+1, \dots, m_2, \dots, m_3, \dots, l) \tag{6.31}$$

This prefactor ensures that the pure gluon amplitudes are correctly reproduced by (6.30). Thus, given the values of each gluon MHV-vertex diagram $\mathcal{A}^{\text{gluons}}_{n,I}$ any NMHV amplitude is simply calculated by applying the ordered string of differential operators associated with the string of external states to the sum over generating functions for each diagram

$$A^{\text{NMHV}}_{n} \;=\; D_1 D_2 \dots D_n\, \mathcal{F}_n\,, \qquad \mathcal{F}_n \;=\; \sum_{I} \mathcal{F}_{n,I}\,. \tag{6.32}$$

Note that this construction automatically produces the correct relative sign of diagrams in the MHV-vertex decomposition.

The sum $\mathcal{F}_n$ of generating functions of the MHV-vertex diagrams is the generating function for the whole NMHV amplitude. Each term $\mathcal{F}_{n,I}$ is a sum of products of 12 distinct η's, so an NMHV n-point amplitude is calculated by applying the appropriate 12th order differential operator composed of n factors from the list (3.3). Each distinct NMHV process corresponds to a particular 12th order operator and conversely. The number of distinct NMHV processes is thus the number of partitions of 12 with $n_{\max} \leq 4$ which is 34. For each given set of particles, i.e. for each process, there are several independent amplitudes in which their order is permuted.[18] For processes with $n < 12$ external particles, the total

[18] In addition there are usually inequivalent assignments of SU(4) indices which give independent amplitudes.

number of NMHV processes is < 34 because one must count only partitions of length $\leq n$. For this reason there are 'only' 18 distinct 6-point processes. Each of these may have several inequivalent assignments of SU(4) indices.

To gain further confidence in the use of the generating function (6.30) let's examine whether the amplitudes obtained from it satisfy SUSY Ward identities. One should bear in mind that these Ward identities need not be satisfied by individual diagrams, but they must hold for the full amplitude. The $\tilde{Q}_a$ Ward identity reads

$$\tilde{Q}_a \tilde{F}_n = 0\,, \tag{6.33}$$

and it is satisfied diagram by diagram, as in the MHV case, because one is again multiplying the $\delta^{(8)}$ function by its own argument. The Q^a Ward identity presents a more interesting situation. Using the two identities

$$Q^a \prod_{b=1}^{4} \sum_{k\epsilon I} \langle P_I\, k\rangle \eta_{kb} \;=\; \sum_{i\epsilon I} [\epsilon\, i] \langle i\, P_I\rangle \prod_{b\neq a} \sum_{j\epsilon I} \langle P_I\, j\rangle \eta_{jb}\,, \qquad \sum_{i\epsilon I} [\epsilon\, i] \langle i\, P_I\rangle \;=\; -\,[\epsilon\, X] P_I^2\,, \tag{6.34}$$

we evaluate

$$\begin{aligned} Q^a \tilde{F}_n &= \frac{1}{V_I} \left(\prod_{i=1}^{n} \langle i\,(i+1)\rangle \right)^{-1} \delta^{(8)}\left(\sum_{i=1}^{n} |i\rangle \eta_{ia} \right) Q^a \prod_{b=1}^{4} \sum_{j\epsilon I} \langle P_I\, j\rangle \eta_{jb} \\ &= -[\epsilon\, X]\, P_I^2\, \frac{1}{V_I} \left(\prod_{i=1}^{n} \langle i\,(i+1)\rangle \right)^{-1} \delta^{(8)}\left(\sum_{i=1}^{n} |i\rangle \eta_{ia} \right) \prod_{b\neq a} \sum_{j\epsilon I} \langle P_I\, j\rangle \eta_{jb}\,. \end{aligned} \tag{6.35}$$

This shows that each diagram contributing to a given amplitude vanishes if we choose $X_\alpha \sim \epsilon_\alpha$. However, provided that the amplitude vanishes for large z, the MHV-vertex formalism ensures that the sum of these diagrams is independent of the reference spinor X_α, so the full amplitude will satisfy the Q^a Ward identity.

At first sight we now seem to be in the same position as we were in the analysis of MHV amplitudes. The full NMHV sector is determined by the values of the diagrams for the n-gluon amplitudes, and all amplitudes satisfy SUSY Ward identities. However, there is an important difference. In the MHV sector there is a unique set of amplitudes which satisfy the Ward identities and agree with the top A_n. In the NMHV sector it is not sufficient to reproduce only A_n; additional input is required.[19] However, the MHV-vertex decomposition contains additional dynamical input, namely the correct analyticity and factorization properties, so we can be confident that it generates the right amplitudes — provided that there is no contribution from infinity in the recursion relations.

6.2.1 NMHV spin factors

We will illustrate the use of the generating function (6.30) and the calculation of spin factors through an example, which will also be relevant for our examination of the large z behavior of NMHV amplitudes in $\mathcal{N} = 4$ SYM theory.

[19] Relations from $\mathcal{N} = 2$ Ward identities were used recently in [29] to simplify the calculation of 6-gluon amplitudes in open string theory.

Consider the six scalar[20] amplitude

$$\big\langle B^{12}(1)\, B^{13}(2)\, B^{14}(3)\, B^{23}(4)\, B^{24}(5)\, B^{34}(6)\big\rangle\,. \tag{6.36}$$

The recursion relations following from a shift of lines 1,2,3 consists of a sum over the six diagrams drawn in figure 5. Each of these diagrams is the product of the result for 6-gluons times a *spin factor* obtained by applying the external state derivatives to the generating function (6.30) and dividing by $\langle m_1 P_I\rangle^4\langle m_2\, m_3\rangle^4$. The spin factor encodes the state dependence of the amplitude. It is a ratio of products of angle brackets and is homogeneous in P_I.

In practice it is simplest to compute the spin factor from the product $\delta^{(8)}(L)\,\delta^{(8)}(R)$ in (6.24) and define it precisely for a generic diagram as

$$\mathcal{S}_I \equiv \mathrm{sign}(I)\,\Big(D_{l+1}\dots D_k\, D_I\, \delta^{(8)}(L)\Big)\,\Big(D_{I'} D_{k+1}\dots D_l\, \delta^{(8)}(R)\Big)/\big(\langle m_i\, P_I\rangle^4\langle m_2\, m_3\rangle^4\big)\,. \tag{6.37}$$

If the diagram is non-vanishing there is a unique choice of the operators D_I and $D_{I'}$ for the internal line which produces the result. The derivative operation is equivalent to a Wick contraction algorithm based on (3.12).

The example of the 6-scalar amplitude (6.36) will make things clear. Consider the 12-pole diagram. Using the notation $\partial_i^a = \partial/\partial\eta_{ia}$ and $\partial_I^a = \partial/\partial\eta_{Ia}$, the derivatives applied to $\delta^{(8)}(L)$ are

$$\partial_1^1\partial_1^2\;\partial_2^1\partial_2^3\;\partial_I^1\partial_I^2\partial_I^3\partial_I^4\,. \tag{6.38}$$

The total derivative order must be 8, so we included the unique internal line derivative of order 4. There is no way to make 4 non-vanishing Wick contractions among these derivatives so the 12-pole diagram vanishes. The same is true for the 23-pole diagram. For the 34 pole diagram we write the string of derivatives

$$\begin{aligned}\mathcal{S}_{34} &= (\partial_3^1\partial_3^4\;\partial_4^2\partial_4^3\;\partial_I^1\partial_I^2\partial_I^3\partial_I^4)\times(\partial_5^2\partial_5^4\;\partial_6^3\partial_6^4\;\partial_1^1\partial_1^2\;\partial_2^3\partial_2^4)/\big(\langle 3\,P_{34}\rangle^4\langle 1\,2\rangle^4\big) \\ &= -\frac{\langle 4\,P_{34}\rangle^2}{\langle 3\,P_{34}\rangle^2}\frac{\langle 1\,5\rangle\langle 2\,6\rangle\langle 5\,6\rangle}{\langle 1\,2\rangle^3} \\ &= -\frac{[3X]^2}{[4X]^2}\frac{\langle 1\,5\rangle\langle 2\,6\rangle\langle 5\,6\rangle}{\langle 1\,2\rangle^3}\,.\end{aligned} \tag{6.39}$$

In the first line we chose the unique 4th order internal derivative which describes the emission of a negative helicity gluon from the left vertex and subsequent absorption as a positive helicity gluon on the right. The second line gives the unique non-vanishing Wick contraction. It exemplifies the general feature that the spin factor is a homogeneous function of angle brackets and also homogeneous in $|P_I\rangle$. In the last line we have used $\langle 4P_{34}\rangle = \langle 4\,3\rangle[3X]$ and a similar equality. We invite readers to compute the remaining 3

[20]We choose this particular configuration of three different "particles" and their "anti-particles" because it is the gauge theory analogue of a 6-scalar amplitude in $\mathcal{N}=8$ supergravity which we will study in detail in section 6.3.2.

non-vanishing spin factors:

$$\begin{aligned}\mathcal{S}_{61} &= -\frac{\langle 6P_{61}\rangle^2}{\langle 1P_{61}\rangle^2}\frac{\langle 24\rangle\langle 35\rangle\langle 45\rangle}{\langle 23\rangle^3} \\ &= -\frac{[1X]^2}{[6X]^2}\frac{\langle 24\rangle\langle 35\rangle\langle 45\rangle}{\langle 23\rangle^3}, \end{aligned} \tag{6.40}$$

$$\begin{aligned}\mathcal{S}_{612} &= \frac{\langle 35\rangle\langle 45\rangle\langle 4P_{612}\rangle}{\langle 3P_{612}\rangle^3}\frac{\langle 26\rangle\langle 1P_{612}\rangle\langle 6P_{612}\rangle}{\langle 12\rangle^3} \\ &= \frac{\langle 4|3+5|X]\langle 1|6+2|X]\langle 6|1+2|X]}{\langle 3|4+5|X]}\frac{\langle 26\rangle\langle 35\rangle\langle 45\rangle}{\langle 12\rangle^3}, \end{aligned} \tag{6.41}$$

$$\begin{aligned}\mathcal{S}_{234} &= \frac{\langle 15\rangle\langle 56\rangle\langle 6P_{234}\rangle}{\langle 1P_{234}\rangle^3}\frac{\langle 24\rangle\langle 3P_{234}\rangle\langle 4P_{234}\rangle}{\langle 23\rangle^3} \\ &= \frac{\langle 6|1+5|X]\langle 3|2+4|X]\langle 4|2+3|X]}{\langle 1|5+6|X]^3}\frac{\langle 15\rangle\langle 24\rangle\langle 56\rangle}{\langle 23\rangle^3}. \end{aligned} \tag{6.42}$$

We have checked numerically that the sum of the four non-vanishing diagrams is independent of the reference spinor $|X]$, and that the amplitude vanishes for large z under a subsequent shift of lines 1,2,3.

6.2.2 Large z behavior in $\mathcal{N}=4$ gauge theory

Since the NMHV generating function (6.32) is based on the recursion relations obtained from the 3-line shift (6.6), it requires that shifted amplitudes vanish as $z \to \infty$. Tree amplitudes always behave as the power law $z^{-\Delta}$ with integer Δ, so we require $\Delta > 0$ for all amplitudes in order to use the NMHV generating function with confidence. The exponent Δ depends on the spin of external states. We now discuss evidence that $\Delta > 0$ for all n-point NMHV amplitudes in $\mathcal{N}=4$ SYM theory.

Each diagram in the MHV-vertex expansion can be written as a spin factor $\mathcal{S}_I$ times the pure gluon diagram. Under a shift of the 3 negative helicity lines, the pure gluon diagram goes to zero for large z at least as fast as $1/z^4$. To see this recall that the gluon diagram is the product of two Parke-Taylor amplitudes and the internal propagator,

$$\begin{aligned}&\frac{\langle m_1\hat{P}_I\rangle^4}{\langle k\hat{P}_I\rangle\langle \hat{P}_I\,(l+1)\rangle(\dots)}\frac{1}{P_I^2}\frac{\langle m_2\, m_3\rangle^4}{\langle l\hat{P}_I\rangle\langle \hat{P}_I\,(k+1)\rangle(\dots)} \\ &\quad= \frac{\langle m_1|P_I|X]^4}{\langle k|P_I|X]\langle l+1|P_I|X](\dots)}\frac{1}{P_I^2}\frac{\langle m_2\, m_3\rangle^4}{\langle l|P_I|X]\langle k+1|P_I|X](\dots)}. \end{aligned} \tag{6.43}$$

The factors in $(\dots)$ do not involve P_I and are not relevant to our argument, in which we perform another 3-line shift of the lines m_i, this time with a new arbitrary reference spinor $|Y]$. The only factors that can shift are those that involve the momentum P_I. Specifically $\langle i|P_I|X]$ shifts, except when $i = m_1$, and the propagator denominator P_I^2 shifts. Simple power counting in (6.43) then shows that for large z the diagram goes as $1/z^5$ when $k,(l+1) \neq m_1$ and as $1/z^4$ otherwise.

The spin factor denominator $\langle m_1 P_I\rangle^4\langle m_2\, m_3\rangle^4$ does not shift. The numerator contains a fourth order product of angle brackets containing $|P_I\rangle$, so under the new $|Y]$-shift, $\mathcal{S}_I$ can at most grow as z^4. Thus, for any 3-line shift the most divergent behavior possible for any diagram is order $O(1)$. However, any NMHV amplitude carries a total of 12 SU(4) indices, and group invariance requires that each distinct index 1, 2, 3 or 4 must occur exactly 3 times among the n external lines. Thus it is always possible to shift 3 lines which have at least one common index, say the index 1. In every MHV partition of the amplitude that same index must also appear on the internal line of the sub-amplitude containing the line m_1. Thus at least one unshifting factor $\langle m_1 P_I \rangle$ occurs in the numerator of every spin factor. Hence, if one chooses a shift with at least one common SU(4) index, every diagram will vanish at least as fast as $1/z$.

This argument strongly supports the conjecture that all NMHV amplitudes of the $\mathcal{N} = 4$ theory are constructible by the MHV-vertex method, but it does not *prove* it. To eliminate the possibility of a contribution from infinity which could invalidate the recursion relations, one would need to determine the asymptotic powers of NMHV amplitudes without using the form of the expansion itself.[21]

Because the general argument does not quite reach its goal, we looked for additional evidence through a numerical study. We have written a Mathematica code which for given SU(4) indices of the external states calculates the MHV-vertex decomposition for any 6-point NMHV amplitude of the $\mathcal{N} = 4$ SYM theory. With this program we have calculated many NMHV amplitudes, and in all cases we have found that there exists at least one 3-line shift such that the associated sum of MHV-vertex diagrams is independent of the reference spinor $|X]$. This is further evidence that there exists a “good” 3-line shift with associated valid recursion relations for any 6-point NMHV amplitude of $\mathcal{N} = 4$ SYM theory.

Large z for the six scalar amplitude in $\mathcal{N} = 4$ SYM theory. In a Feynman diagram analysis, the polarization vectors of the 3 negative helicity gluons provide the power z^{-3} at large z. When the gluons are replaced by scalars, this asymptotic damping is lost. This suggests that the least favorable asymptotic behavior occurs for external scalars which have neither polarization vectors nor external spinors. With this in mind we discuss the large z behavior of the gauge theory 6-point amplitude $\langle B^{12}(1)\, B^{13}(2)\, B^{14}(3)\, B^{23}(4)\, B^{24}(5)\, B^{34}(6)\rangle$ whose spin factors were calculated in section 6.2.1.

Using the explicit results for the four non-vanishing spin factors given in (6.39) and (6.40), one readily sees that under a subsequent $|Y]$-shift of lines 123, the large z behavior is

$$\mathcal{S}_{34} \sim z^2\,, \quad \mathcal{S}_{61} \sim z^2\,, \quad \mathcal{S}_{612} \sim z^3\,, \quad \mathcal{S}_{234} \sim z^3\,. \tag{6.44}$$

Each spin factor $\mathcal{S}_I$ must be multiplied by the value of the corresponding diagram in figure 5 for the 6-gluon process. Since the leading behavior of each of these gluon diagrams is $1/z^4$, the leading contribution to the 6-scalar amplitude comes from the 3-particle diagrams and is $1/z$. There is no cancellation, so the falloff of the full amplitude is $1/z$. We also checked the behavior of the 6-scalar amplitude under 2-line shifts. The large z behavior is $1/z^2$ for a $[1,3\rangle$ shift, but $O(1)$ for a $[1,2\rangle$ shift and $O(z)$ for a $[1,6\rangle$ particle-antiparticle shift.

[21] It has subsequently been proven in [19] that there always exists a valid 3-line shift for any NMHV amplitude of $\mathcal{N} = 4$ SYM theory.

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We have also used the generating function to construct all other 6-scalar amplitudes with different configurations of SU(4) indices. In every case the sum of MHV-vertex diagrams is independent of $|X]$.

6.3 Generating function for NMHV amplitudes in $\mathcal{N} = 8$ supergravity

At the formal level, it is not difficult to extend the construction of the previous section to supergravity, but the issue of large z behavior will become acute. The extension is based on the MHV-vertex formalism for n-graviton amplitudes of [17] which we have discussed in section 6.1.

The NMHV sector of $\mathcal{N} = 8$ supergravity consists of all amplitudes related to the top n-graviton amplitude $M_n(1^-, 2^-, 3^-, 4^+, \ldots, n^+)$ by SUSY Ward identities. In analogy with the $\mathcal{N} = 4$ theory, the practical definition of this sector is that it contains amplitudes for all sets of external particles for which the associated differential operator constructed from products of n operators from the correspondence (3.16) is of total order 24. We will justify this definition below.

If an NMHV amplitude vanishes at large z under the 3-line shift, it also obeys a recursion relation (equivalently, it has a valid MHV-vertex decomposition) similar to (6.19). The contribution of a generic diagram corresponding to figure 6 has the same structure as the gauge theory formula (6.24) and can be written as

$$\mathcal{M}_{n,I} \equiv \text{sign}(I)\, \frac{M_{n_1}(l+1, \ldots, \hat{m}_1, \ldots, k, -\hat{P}_I)}{\langle m_1\, \hat{P}_I\rangle^8} \frac{1}{s_I} \frac{M_{n_2}(\hat{P}_I, k+1, \ldots, \hat{m}_2, \ldots, \hat{m}_3, \ldots, l)}{\langle m_2\, m_3\rangle^8} \\ \times \Big(\mathcal{D}_{l+1} \ldots \mathcal{D}_k\, \mathcal{D}_I\, \delta^{(16)}(L)\Big) \Big(\mathcal{D}_{I'} \mathcal{D}_{k+1} \ldots \mathcal{D}_l\, \delta^{(16)}(R)\Big)\,, \qquad (6.45)$$

The arguments L and R of the δ-functions are given in (6.25), and the detailed form of the shifted sub-amplitudes M_{n_1} and M_{n_2} are given in (6.21) and (6.23). Note that the derivatives acting on each δ-function are of total order 16.

We follow the same steps used in the gauge theory case to obtain the generating function for this diagram

$$\tilde{\Omega}_{n,I} = \frac{M_{n_1}(l+1, \ldots, \hat{m}_1, \ldots, k, \ldots, -\hat{P}_I)}{\langle m_1\, \hat{P}_I\rangle^8} \frac{1}{s_I} \frac{M_{n_2}(\hat{P}_I, k+1, \ldots, \hat{m}_2, \ldots, \hat{m}_3, \ldots, l)}{\langle m_2\, m_3\rangle^8} \\ \times \delta^{(16)}\Big(\sum_{i=1}^{n} |i\rangle \eta_{iA}\Big) \prod_{B=1}^{8} \sum_{j=l+1}^{k} \langle \hat{P}_I\, j\rangle \eta_{jB}\,. \qquad (6.46)$$

The generating function for the full amplitude is then $\sum_I \tilde{\Omega}_{n,I}$, where the sum runs over all $3(2^{n-3} - 1)$ internal pole channels.

In the process of deriving (6.46) 8 internal derivatives $\mathcal{D}_I \mathcal{D}_{I'}$ were converted to integrals and eliminated. The remaining external line derivatives $\mathcal{D}_1 \cdots \mathcal{D}_n$ which one applies to $\tilde{\Omega}_I$ to (re)obtain the diagram (6.45) are of total order $32 - 8 = 24$. Thus each distinct n-point NMHV process in $\mathcal{N} = 8$ supergravity corresponds to a particular 24th order differential operator composed of n factors from the correspondence (3.16). The number of distinct

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NMHV amplitudes is the number of partitions of 24 with $n_{\max} \leq 8$ which is 919. One needs n-point functions with $n \geq 24$ to realize this maximum number. For $n < 24$, the partition length must be $\leq n$, so there are fewer types of NMHV amplitudes. For $n = 6$ there are 151 distinct processes.

The structure of the formula (6.46) is analogous to (6.30) in gauge theory. In gauge theory the dynamical function which multiplies the Grassmann factors is the value of the n-gluon diagram divided by $\langle \ldots \rangle^4 \langle \ldots \rangle^4$ for the negative helicity lines of the diagram. In supergravity it is the n-graviton diagram divided by $\langle \ldots \rangle^8 \langle \ldots \rangle^8$ for the negative helicity lines. One significant difference is that one must insert the correctly shifted anti-holomorphic spinors $|\hat{m}_i]$ for the 3 distinguished negative helicity gravitons.

The contribution of each diagram to a particular amplitude of interest is obtained by applying the appropriate order 24 product of external state derivatives to the generating function. Each diagram has its own spin factor which is obtained in this way. Our experience indicates that it is easiest to calculate the spin factor by applying the derivatives to the product of δ-functions for each sub-amplitude. Thus, in analogy with (6.37), the spin factor for a MHV-vertex diagram in supergravity is defined by

$$\mathcal{S}_I \equiv \Big(\mathcal{D}_{l+1} \ldots \mathcal{D}_k \, \mathcal{D}_I \, \delta^{(16)}(L) \Big) \Big(\mathcal{D}_{I'} \mathcal{D}_{k+1} \ldots \mathcal{D}_l \, \delta^{(16)}(R) \Big) / (\langle m_i \, P_I \rangle^8 \langle m_2 \, m_3 \rangle^8) \,. \qquad (6.47)$$

The products include a pair of internal line derivatives $\mathcal{D}_I$ and $\mathcal{D}_{I'}$ which are uniquely determined by SU(8) covariance and the fact that the total order of derivatives on each $\delta^{(16)}$ must equal 16. The simplest way to calculate uses the Wick contractions of (3.12). The spin factors of some diagrams may vanish, implying that the diagram makes no contribution to the amplitude.

In the previous section, we argued that amplitudes obtained from the NMHV generating function in $\mathcal{N} = 4$ SYM theory satisfy the SUSY Ward identities. The same argument applies to supergravity, so it is clear that the amplitudes obtained from (6.46) satisfy the Ward identities of $\mathcal{N} = 8$ supergravity if the MHV-vertex expansion is valid.

The validity of the generating function relies on the vanishing of the shifted amplitudes for large z. While there is evidence that such "good" shifts can always be found for NMHV amplitudes in $\mathcal{N} = 4$ SYM theory, we have found explicit counter-examples in $\mathcal{N} = 8$ supergravity. We will discuss large z behavior in section 6.3.2, including several examples and the lessons they teach us.

6.3.1 Factorization

In section 3.3 we discussed the factorization of the MHV generating function for supergravity. Factorization ensures that MHV amplitudes are compatible with the operator map (1) and that all symmetries are consistently implemented. A similar factorization with similar consequences holds for the generating function of each MHV-vertex diagram in the NMHV sector. We observe that the Grassmann terms in (6.46) factor into a product of two factors of the analogous terms for gauge theory in (3.1), one each for the L and R gauge theory factors in the map (1.1). Thus the supergravity generating function for each diagram can be rewritten as

$$\tilde{\Omega}_I = \frac{M_{n_1}(l+1,\ldots,\hat{m}_1,\ldots,k,\ldots,-\hat{P}_I)}{\langle m_1\,\hat{P}_I\rangle^8}\frac{1}{s_I}\frac{M_{n_2}(\hat{P}_I,k+1,\ldots,\hat{m}_2,\ldots,\hat{m}_3,\ldots,l)}{\langle m_2\,m_3\rangle^8} \tag{6.48}$$
$$\times\left(\delta^{(8)}\Big(\sum_{i=1}^n |i\rangle\eta_{ia}\Big)\prod_{b=1}^4\sum_{j=l+1}^k\langle\hat{P}_I\,j\rangle\eta_{jb}\right)\times\left(\delta^{(8)}\Big(\sum_{i=1}^n |i\rangle\eta_{ir}\Big)\prod_{c=5}^8\sum_{j=l+1}^k\langle\hat{P}_I\,j\rangle\eta_{jc}\right).$$

We see that the spin factors for supergravity amplitudes are products of spin factors for the appropriate gauge theory amplitudes. This means that SU(8) and SUSY $\tilde{Q}_A$ Ward identities, which hold separately for each term in the MHV-vertex expansion, are satisfied on the gauge theory side of the map (1.1). The same conclusion holds for SUSY Q^A Ward identities after summation of all contributing diagrams, provided that the sum is independent of $|X]$.

6.3.2 Large z behavior of NMHV amplitudes in $\mathcal{N}=8$ supergravity

As in gauge theory, the shifted tree amplitudes of $\mathcal{N}=8$ supergravity behave as $z^{-\Delta}$ for large z, and the validity of the generating function requires $\Delta > 0$. The arguments that gave evidence for this in gauge theory do not carry over to supergravity. Indeed, we will present explicit counter-examples, namely NMHV amplitudes of $\mathcal{N}=8$ supergravity for which no 3-line shift gives $\Delta > 0$. As in section 6.2.2 we begin the discussion by determining the large z behavior of typical diagrams in the MHV-vertex expansion. Each diagram is the product of the result for n external gravitons times a spin factor. The general discussion will tell us what to expect at large z, and the actual behavior will then be illustrated in several examples.

General discussion. Our first task is to ascertain the large z asymptotics of a typical diagram for the 6-graviton NMHV amplitude $M_6(1^-,2^-,3^-,4^+,5^+,6^+)$.[22] The formulas (6.21) and (6.23) contain most of the information needed to extract the power of z obtained from a further scaling of $|1],|2],|3]$ in a generic direction $|Y]$ in spinor space. However 2-particle pole diagrams, which contain the factor M_3, must be examined separately. It is not difficult to obtain the following information about the large z behavior:

i. 2-particle pole diagrams for external gravitons with $--$ helicity vanish at the rate $1/z^7$.

ii. 2-particle poles with graviton helicity $-+$ vanish more slowly, namely at the rate $1/z^5$.

iii. 3-particle pole diagrams, necessarily with $--+$ and $-++$ helicities in each subamplitude, vanish as $1/z^6$.

These estimates apply to each individual diagram. It is possible that there are cancellations among diagrams, so that the full amplitudes actually fall off faster.

[22] n-point amplitudes with $n>6$ are briefly discussed in section 6.3.5.

The MHV-vertex decomposition of $M_6(1^-,2^-,3^-,4^+,5^+,6^+)$ contains 21 non-vanishing diagrams. Numerical results show that the sum of these diagrams is independent of $|X]$ and vanishes as $1/z^6$ upon a further shift of lines 123. One can see analytically that there is a cancellation among the nine 2-particle pole diagrams with $-+$ helicities. We have also checked that the MHV-vertex method and the KLT formula produce the same result.

For general external states, the MHV-vertex expansion expresses the amplitude as a sum of n-graviton diagrams multiplied by spin factors. See (6.45) and (6.47). The spin factors are readily computed for any given process, but it is useful to have general estimates of their growth rate at large z as an indication of the behavior of the full amplitude. We consider the z-dependence of the spin factors for a shift of 3 chosen lines labelled m_1, m_2, m_3. The key parameter that determines the large z growth rate is the number of SU(8) indices which appear on all three shifted lines. We let $n_{\rm com}$ denote the number of common indices.

For the spin factors of 6-point amplitudes we can prove the following:

A. For diagrams with a 2-particle pole in the m_2m_3 channel, the product of spin factors grows no faster than $z^{8-n_{\rm com}}$.

B. For any 3-particle pole diagram, the maximum growth rate of the product of spin factors is also $z^{8-n_{\rm com}}$.

C. For diagrams with a 2-particle pole in a channel with one shifted line, say m_1 and one unshifted line a, the product of spin factors grows at the rate z^{r_a}, where r_a is the η-count of particle a (defined as the order of the corresponding Grassmann derivative).

The proof of A and B is quite simple. The numerator of the product of spin factors in (6.47) contains a product of 8 brackets $\langle i\, P_I\rangle$ where i denotes any of the 6 external lines. All products except $\langle m_1\, P_I\rangle$ grow linearly with z after a shift. This bracket occurs as $\langle m_1\, P\rangle^{\nu_1}$ in the product of spin factors, so the growth rate of that product is $z^{8-\nu_1}$. We will show that $\nu_1 \geq n_{\rm com}$ which will prove the bound on growth rate stated above. To establish this inequality we refer to figure 6. Lines m_2 and m_3 have at least $n_{\rm com}$ SU(8) indices in common. Therefore these indices cannot appear at the right end of the internal line. The reason is that each sub-amplitude must be an SU(8) singlet, so that the 16 indices either one contains must comprise 8 *distinct matched pairs*. The common indices must then appear on the left end of the internal line, and at least $n_{\rm com}$ of them are shared with line m_1. Thus $\nu_1 \geq n_{\rm com}$ and the proof is finished. (Note that the maximum growth rate $z^{8-n_{\rm com}}$ is actually valid for the spin factors of all diagrams of any n-point NMHV amplitude.)

The proof of C follows from the CFT analogy (see section 5) which gives $D_{m_1}D_aD_I\,\delta^{16}(L) \propto \langle m_1\, P_I\rangle^{8-r_a}$. This leaves $8-(8-r_a)=r_a$ powers of $\langle\cdot P_I\rangle$ that shift in the product of spin factors for the two sub-amplitudes. (This argument also applies to n-point amplitudes).

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The information on the growth of spin factors can now be combined with the estimates of the 6-graviton prefactors to give the asymptotic growth rates of each type of diagram.

i. 2-particle pole diagrams with two shifted lines have prefactors which fall off as $1/z^7$. After multiplying by the (worst case) rate $z^{8-n_{\rm com}}$ for the spin factor we see that the maximum growth rate of the diagram is $O(1)$.

ii. A 2-particle pole diagrams with one shifted and one unshifted line behaves as z^{r_a-5} The growth rate is no worse than $O(1)$ unless $r_a \geq 6$. We can always choose to shift three lines with $r_{m_i} \geq r_a$. Since the total η-count of an NMHV amplitude is 24, the case $r_a = 7$ is then eliminated. The only process with $r_a = 6$ we need concern ourselves with has partition $6+6+6+6$. For this four graviphoton amplitude, it turns out that the potential linear divergences cancel by the same mechanism that the pure graviton amplitude's leading $1/z^5$ terms cancels down to $1/z^6$.

   Thus we never encounter worse than $O(1)$ asymptotics from the 2-particle pole diagrams.

iii. For 3-particle pole diagrams the maximum growth rate is $z^{2-n_{\rm com}}$, so we may encounter linear growth and $O(1)$ behavior.

The bound $z^{8-n_{\rm com}}$ on the growth of spin factors suggests that the optimal behavior at large z can be obtained by shifting 3 lines with the largest value of $n_{\rm com}$. Indeed, for $M_6(1^-2^-3^-4^+5^+6^+)$ the conventional $---$ shift realizes the maximal value $n_{\rm com} = 8$. In our numerical exploration of large z behavior of non-optimal shifts were also included and were instructive. Since the total η-count of an NMHV amplitude is 24 and SU(8) symmetry requires that every index $1,2,\ldots,8$ must appear exactly 3 times among the 6 lines, it is always possible to choose a shift with $n_{\rm com} = 1$. One might suspect that the recursion relation would fail for a shift of lines m_1, m_2, m_3 with $n_{\rm com} = 0$ because a physical pole in the channel $m_1m_2m_3$ would be omitted. However, in an example below we find a valid recursion relation for such a shift.

Let us now turn to the results of our extensive numerical study of NMHV amplitudes in $\mathcal{N} = 8$ supergravity. In turn we will discuss "good" amplitudes for which the sum of diagrams vanishes as $z \to \infty$, "bad" amplitudes in which $O(1)$ behavior occurs in the sum, and "very bad" amplitudes with linear growth. We will argue that both "bad" and "very bad" amplitudes still have valid recursion relations.

"Good" amplitudes. There are numerous examples of NMHV amplitudes in $\mathcal{N} = 8$ supergravity for which there are 3-line shifts with $\Delta > 0$ and hence valid MHV-vertex expansion. The generating function works for these examples just as it did in gauge theory. However we will briefly discuss some examples which reveal interesting regularities.

Consider the amplitudes $\langle b^- \, b^- \, f_1^- \, f_+^1 \, b_+ \, b_+\rangle$, $\langle b^- \, b^- \, b_{12}^- \, b_+^{12} \, b_+ \, b_+\rangle$, $\langle b^- \, b^- \, f_{123}^- \, f_+^{123} \, b_+ \, b_+\rangle$, and $\langle b^- \, b^- \, b_{1234} \, b^{1234} \, b_+ \, b_+\rangle$, in which a graviton pair is replaced, in turn, by a pair of gravitini, graviphotons, graviphotini, and scalars. All amplitudes have 18 contributing diagrams whose sum is independent of $|X]$. Let's label each particle by its spin s, with $s = 2$ for the graviton, $s = 3/2$ for the gravitino, etc. It is interesting to observe a simple pattern for the spin factors of each diagram as s decreases. For any given diagram $\mathcal{A}_{n,I}$ let us simply denote by $\mathcal{S}_I$ its spin factor in the gravitino amplitude. For other cases the same diagram has the spin factor $\mathcal{S}_I^{(4-2s)}$. Furthermore the amplitude with the spin s pair vanishes as $1/z^{2s+2}$ as $z \to \infty$ under a further 123 shift. The pattern in the NMHV sector is similar to what occurred in the analogous set of MHV amplitudes in (2.36).

Another example of a "good" amplitude is the scalar amplitude

$$\big\langle b^{1234}\, b^{1234}\, b^{1234}\, b^{5678}\, b^{5678}\, b^{5678} \big\rangle\,, \tag{6.49}$$

whose external states are three identical scalars b^{1234} and their conjugates b^{5678}. A KLT calculation shows that the amplitude (6.49) has a $1/z^2$ falloff for large z under shifts that do not involve conjugate scalars,[23] such as the 123-shift, and the resulting recursion sum of 18 MHV-vertex diagrams therefore gives the correct amplitude and is indeed, as shown in our numerical work, independent of $|X]$.

"Bad" amplitudes. It was an unwelcome discovery that by a mere change of the SU(8) labels in the six scalar amplitude, one finds an amplitude

$$\big\langle b^{1234}\, b^{1358}\, b^{1278}\, b^{5678}\, b^{2467}\, b^{3456} \big\rangle\,, \tag{6.50}$$

for which the sum of MHV-vertex diagrams depends on $|X]$. This result appears to be unacceptable, so we proceed to study it further, specifically by an independent construction using the KLT formula. (The method is explained in more detail below.) The KLT result is valid for general complex momenta and we can explore the large z behavior by making various 3-line shifts numerically. The amplitude consists of three pairs of conjugate scalars (e.g. b^{1234} and b^{5678}) and has only two types of 3-line shifts, namely shifts involving no conjugate pairs — such as a 123-shift — and shifts that involve a conjugate pair of scalars (e.g. a 124-shift). The former give $O(1)$ for large z and the latter $O(z^2)$ (since the 3-lines do not share a common index, $n_{\rm com} = 0$). Thus there are no 3-line shifts of the six scalar amplitude (6.50) that give large z falloffs faster then $O(1)$, and we therefore categorize it as a "bad" amplitude.

At first sight, a MHV-vertex decomposition based on 3-line recursion relations seems to be impossible for "bad" amplitudes. However, 3-line shifts with $O(1)$ asymptotics can still be used to derive a recursion formula if the reference spinor $|X]$ is suitably chosen. To explain our approach to "bad" amplitudes we start with the example of the pure gluon amplitude in gauge theory:

Example: $O(1)$ shifts in gauge theory and the role of $|X]$. The gluon amplitude $A_6(1^-,2^-,3^-,4^+,5^+,6^+)$ is well known and we have already seen that it can be calculated with the MHV-vertex method associated with a shift of the three negative helicity lines. But it is illustrative to consider the large z behavior of other 3-line shifts of the amplitude.

[23] If the lines shifted involve a scalar and its conjugate then the shifted amplitude grows as z^2 for large z.

This can be done numerically and we find

$$\begin{aligned}
\langle \hat{-}\hat{-} - \hat{+} + + \rangle &\sim \frac{1}{z}\,, & \langle - - - \hat{+}\hat{+}\hat{+} \rangle &\sim O(1)\,, \\
\langle \hat{-} - \hat{-} + \hat{+} + \rangle &\sim \frac{1}{z}\,, & \langle \hat{-}\hat{-} - + + \hat{+} \rangle &\sim O(1)\,, \\
\langle - \hat{-} - \hat{+} + \hat{+} \rangle &\sim \frac{1}{z}\,, & \langle - - \hat{-}\hat{+}\hat{+} + \rangle &\sim O(1)\,.
\end{aligned} \tag{6.51}$$

Cauchy's theorem gives valid recursion relations for the first three types of shifts for which the amplitude vanishes as $z \to \infty$. In all three cases, the sum of MHV-vertex diagrams is independent of $|X]$.

The shifts which give $O(1)$ asymptotics also give rise to valid recursion relations, but only for the special values of $|X]$ for which the $O(1)$-contribution vanishes. This condition is a polynomial equation in $|X]$, and for each of the three cases above, we have found the roots $|X_q]$ numerically. The asymptotic $O(1)$ term vanishes at each root, so the conditions needed to use Cauchy's theorem are satisfied. Indeed we have verified that precisely for these values of $|X]$ the recursion sum of MHV-vertex diagrams agrees with the result from the $- - -$ shift and thus gives the correct 6-gluon amplitude. For any other values of $|X]$ the recursion sum is invalid; the contribution from "infinity" in Cauchy's theorem is missing.

This result is particularly striking for the last two shifts which each involve 3 lines with $n_{\text{com}} = 0$. Thus there is no diagram which contains the gluon pole in the channel of the shift. Nevertheless this pole is reproduced by the other diagrams at the special values $|X_q]$.

We have found that the situation is similar for all "bad" NMHV 6-point amplitudes in $\mathcal{N} = 8$ supergravity. This leads to a modified criterion for the validity of the MHV-vertex method which we now summarize:

— If a shifted amplitude goes to zero for any $|X]$, then the sum of MHV-vertex diagrams resulting from that 3-line shift must be independent of $|X]$ and gives a correct expression for the amplitude.

— If a shifted amplitude does not go to zero for all $|X]$, then the sum of MHV-vertex diagrams resulting from that 3-line shift depends on $|X]$. Nonetheless, the corresponding MHV-vertex method gives the correct result, provided that $|X]$ is chosen to eliminate the $O(1)$-term which is the residue of the simple pole at infinity.

With these rules we can apply the generating function to the large number of the NMHV amplitudes in the $\mathcal{N} = 8$ theory whose best shifts give $O(1)$ for large z. We have tested this procedure in several examples, including the six scalar amplitude (6.50). We

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first calculate the amplitude using the KLT formula,

$$\begin{aligned}\big\langle a(1)\,a(2)\,a(3)\,a(4)\,a(5)\,a(6)\big\rangle = &\Big\{ s_{34}\, s_{16} \big\langle A(1)\, A(2)\, A(3)\, A(4)\, A(5)\, A(6)\big\rangle \\ &\times\Big[s_{15}\, \big\langle \tilde{A}(1)\, \tilde{A}(3)\, \tilde{A}(4)\, \tilde{A}(2)\, \tilde{A}(6)\, \tilde{A}(5)\big\rangle \\ &+(s_{15}+s_{56})\, \big\langle \tilde{A}(1)\, \tilde{A}(3)\, \tilde{A}(4)\, \tilde{A}(2)\, \tilde{A}(5)\, \tilde{A}(6)\big\rangle \Big]\Big\} \\ &+\mathcal{P}(4,5,6)\,. \end{aligned} \tag{6.52}$$

The a's can be annihilation operators for any states of the $\mathcal{N}=8$ theory, and A and $\tilde{A}$ denote the decomposition of the a operators under the map (1). We use the NMHV generating function of section 6.2 to calculate each gauge theory amplitude.

Different ways to split SU(8) $\to$ SU(4) $\times$ SU(4) result in different decompositions $a = A \otimes \tilde{A}$, but the r.h.s. of (6.52) must give the same result for the supergravity amplitude. Calculating the supergravity amplitudes from different KLT decompositions provides a useful check on the correctness of the result.

Next we perform a 3-line shift of the supergravity amplitude (6.52) with an arbitrary reference spinor $|X]$. The $O(1)$ term for large z is a function of $|X]$, $f = f(|X])$. Setting $f(|X]) = 0$ gives a polynomial equation in $|X]$, and its roots $|X_q]$ make the recursion relation valid. For the six scalar amplitude (6.50) there are six solutions $|X_q]$. We then compute the MHV vertex expansion for the same shift, and evaluate it for $|X] = |X_q]$. The sum of diagrams always agrees with the KLT result and thus confirms the validity of the procedure.

The six scalar amplitude (6.50) is not the only amplitude whose best 3-line shift gives $O(1)$ for large z. We list here a selection which illustrates that the $O(1)$ behavior occurs in a variety of different cases, not necessarily involving scalars.

$$\begin{aligned} &\big\langle b^-_{78}\, b^-_{56}\, b^{1567}\, b^{2578}\, b^{36}_+\, b^{48}_+\big\rangle \\ &\big\langle b^-\, b^{1234}\, b^{1567}\, f^{258}_+\, f^{368}_+\, b^{47}_+\big\rangle \\ &\big\langle f^-_{678}\, b^{1358}\, b^{1278}\, b^{5678}\, b^{2467}\, f^{346}_+\big\rangle \\ &\big\langle f^-_{678}\, f^-_{458}\, f^-_{235}\, f^{268}_+\, f^{578}_+\, f^{345}_+\big\rangle \\ &\big\langle b^-_{12}\, b^-_{34}\, b^-_{56}\, b^-_{78}\, b_+\, b_+\big\rangle \end{aligned} \tag{6.53}$$

We have calculated each of these amplitudes using the KLT formula (6.52), determined $|X^*]$ such that the asymptotic $O(1)$ vanished, and verified numerically that the generating function gives the correct values for $|X] = |X^*]$.[24]

We have attempted to test how many of the 151 partitions of distinct 6-point NMHV processes contain "bad" amplitudes. A preliminary count gives 73; this based on a scan of different SU(8) index structures and tests of large z asymptotics of the diagrams of the MHV-vertex expansion associated with all possible 3-line shifts.

[24] The order of the polynomial $f(|X])$ typically varies in the range 2-8.

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"Very bad" amplitudes. We finally turn our attention to the "very bad" amplitudes. With the help of Mathematica we have analyzed which 6-point NMHV amplitudes have the property that no 3-line shifts give better behavior than $O(z)$ for large z. We find that there are only two such amplitudes, namely

$$\big\langle f^{-}_{678}\, b^{2568}\, b^{3478}\, b^{4578}\, b^{1367}\; f^{126}_{+}\big\rangle\,, \qquad \big\langle f^{-}_{678}\, f^{-}_{458}\, f^{-}_{238}\, b^{2468}\, b^{3578}\; f^{8}_{+}\big\rangle\,. \tag{6.54}$$

We have computed both amplitudes using the KLT relations and confirmed that they grow linearly in z for large z under *any* one of the twenty possible different 3-line shifts. The linear growth term does not invalidate the recursion relation, but there is a contribution from the subleading $O(1)$-term. As in the case of "bad" amplitudes, we can choose $|X]$ to set this term to zero, and the MHV-vertex expansion is then found numerically to agree with the KLT result.

One may wonder how the Ward identities can accommodate amplitudes with different large z behaviors; this is the subject of the next section.

6.3.3 Supersymmetric Ward identities and large z

As discussed in section 2.4, SUSY Ward identities in the NMHV sector always relate sets of four amplitudes. To see this explicitly in the $\mathcal{N}=4$ and $\mathcal{N}=8$ theories, one must choose specific values of the flavor indices. The generic form of any NMHV SUSY Ward identity is therefore

$$0 = \langle\epsilon\, i_1\rangle\, A_1 + \langle\epsilon\, i_2\rangle\, A_2 + \langle\epsilon\, i_3\rangle\, A_3 + \langle\epsilon\, i_4\rangle\, A_4\,, \tag{6.55}$$

where $i_k = 1,2,3,4,5$, or 6. There are a variety of possibilities. A given Ward identity can involve only "good" amplitudes, or both "good" and "bad", only "bad", etc. This terminology refers to asymptotic behavior under an optimally chosen shift of each amplitude. To investigate the large z asymptotics of the entire Ward identity, one must use the *same* shift to analytically continue all four amplitudes.

Under any such common shift, we can assume that for large z, the amplitudes behave as $A_i \sim z^{k_i}$ for large z. Without loss of generality, let us assume that $k_1 \geq k_2 \geq k_3 \geq k_4$. Start by setting $|\epsilon\rangle = |x_1\rangle$ in (6.55). Then we must have $k_2 = k_3$, because either $k_2 = k_3 = k_4$, or — if $k_3 > k_4$ — then $k_2 = k_3$, so that the leading powers z^{k_2} and z^{k_3} cancel down to z^{k_4}. Likewise, we determine from $|\epsilon\rangle = |i_3\rangle$ that $k_1 = k_2$. We conclude that the SUSY Ward identity (6.55) restricts the powers of the leading z-behaviors to be $k_1 = k_2 = k_3 \geq k_4$. Thus for each shift, the four amplitudes in the Ward identity can at most involve two different large z powers, and the slowest falloff must occur at least thrice.

Let's see how this works in practice. Consider the Ward identity

$$\begin{aligned}
0 &= \big\langle[\tilde{Q}_6,\, b^{-}_{78}\, b^{2568}\, b^{3478}\, b^{4578}\, b^{1367}\; f^{126}_{+}]\big\rangle \\
&= \langle\epsilon\, 1\rangle \big\langle f^{-}_{678}\, b^{2568}\, b^{3478}\, b^{4578}\, b^{1367}\; f^{126}_{+}\big\rangle + \langle\epsilon\, 2\rangle \big\langle b^{-}_{78}\; f^{258}_{+}\, b^{3478}\, b^{4578}\, b^{1367}\; f^{126}_{+}\big\rangle \\
&\quad + \langle\epsilon\, 5\rangle \big\langle b^{-}_{78}\, b^{2568}\, b^{3478}\, b^{4578}\; f^{137}_{+}\; f^{126}_{+}\big\rangle + \langle\epsilon\, 6\rangle \big\langle b^{-}_{78}\, b^{2568}\, b^{3478}\, b^{4578}\, b^{1367}\; b^{12}_{+}\big\rangle\,.
\end{aligned} \tag{6.56}$$

We recognize the first amplitude of (6.56) as one of the "very bad" amplitudes (6.54). The three other amplitudes in the Ward identity (6.56) turn out to be just "bad". Under any

3-line shift, the first amplitude give $O(z)$ for large z. Depending on the choice of which three lines are shifted, the Ward identity (6.56) accommodates three different combinations of large z behaviors:

- All four amplitudes grow as $O(z)$ (e.g. 134-shift).
- One amplitude gives $O(1)$ and the three others give $O(z)$. This happens only in 5 cases: the 156-shift gives $O(1)$ for the second amplitude, the 126-shift gives $O(1)$ for the third amplitude and the 124-, 135- and 125-shifts give $O(1)$ for the fourth amplitude.
- The three "bad" amplitudes grow as z^2 for large z while (as always) the "very bad" amplitude grows as z. This occurs when the three shifted lines involve three states in the "bad" amplitudes which do not share a common index.

We have verified this in explicit numerical calculations, with amplitudes computed by the KLT formula. Numerical tests included Ward identities for both the $\tilde{Q}_A$ and Q^A operators, The pattern of large of z asymptotics found here is in complete agreement with the general analysis.

6.3.4 2-line shifts vs. 3-line shifts

We would like to point out some differences — and relationships — between the 2- and 3-line shifts. First of all, the 3-line shifts involve the arbitrary reference spinor $|X]$. The fact that Cauchy's theorem only requires $M_n(z) \to 0$ for $z \to \infty$ for *some* $|X]$, allow us to use the generating function and the MHV-vertex expansion even for amplitudes whose best shifts go as $O(1)$ for large z; the $|X]$ must be chosen such that the $O(1)$ term vanishes. This freedom is clearly not available in the 2-line recursion relations.

An example illustrating the differences between the 2- and 3-line shifts is the "bad" six scalar amplitude (6.50). There are *no* valid 2-line shifts for this amplitude; if a pair of conjugate scalars is shifted, the amplitude grows as z^2 for large z, while if a pair of non-conjugate scalars are shifted, then the large z behavior is $O(1)$. On the other hand, the 123-shift recursion relations give a valid MHV-vertex decomposition of the amplitude for the six special values of $|X]$ for which the $O(1)$-term of the large 123-shift vanishes.

It was pointed out in [17] that the 3-line shifts can be built from three successive 2-line shifts, viz.

$$\begin{aligned}
|\hat{1}] &= |1] + z\,\langle 23\rangle |X]\,, \quad |\hat{X}\rangle = |X\rangle - z\,\langle 23\rangle |1\rangle\,,\\
|\hat{2}] &= |2] + z\,\langle 31\rangle |X]\,, \quad |\hat{X}\rangle = |X\rangle - z\,\langle 31\rangle |2\rangle\,,\\
|\hat{3}] &= |3] + z\,\langle 12\rangle |X]\,, \quad |\hat{X}\rangle = |X\rangle - z\,\langle 12\rangle |3\rangle\,.
\end{aligned} \tag{6.57}$$

The spinor $|X]$ can be chosen as the holomorphic spinor of lines 4, 5, or 6, since the cumulative shift of $|X\rangle$ cancels by the Schouten identity.

The two amplitudes in (6.54) are problematic because they cannot be computed with the MHV-vertex method for any $|X]$. However, there do exist good 2-line shifts for both amplitudes, e.g. the $[1,6\rangle$-shift works for both. (The resulting 2-line recursion relations will involve anti-MHV vertices.)

The existence of (three) valid 2-line shifts does *not* imply that the combined 3-line shift (6.57) is valid. An explicit example of this is provided by the second amplitude $\langle f^-_{678}\, f^-_{458}\, f^-_{238}\, b^{2468}\, b^{3578}\, f^8_+\rangle$ of (6.54), for which all of the twenty possible 3-line shifts give $O(z)$ for large z. Under each of the 2-line shifts $[1,6\rangle$, $[2,6\rangle$, $[3,6\rangle$ this amplitude goes as $1/z$ for large z, but the combined shift is a 3-line shift with $|X] = |6]$, and we know that the amplitude will not go to zero for large z under such a shift. In fact, for the 123-shift with $|X] = |6]$ the amplitude goes to a constant. The reason that one finds $O(1)$ rather than $O(z)$ is that $|X] = |6]$ happens to be one of the solutions to setting $O(z) = 0$ for the 123-shift.

We will see next that there are problems with 3-line shifts even for pure graviton NMHV amplitudes for sufficiently large n.

6.3.5 n-point NMHV graviton amplitudes with $n > 6$

One can estimate the large z falloff of the n-point graviton NMHV amplitude from the large z behavior of the individual diagrams in the MHV-vertex expansion (6.19). The large z asymptotics of the two MHV-vertices can be extracted from the BGK formula, as presented in (6.21) and (6.23). For $n = 6$, the slowest falloff comes from the $+-$ 2-particle pole diagrams and is $1/z^5$. However, due to a cancellation among these diagrams, the falloff of the full amplitude is $1/z^6$.

In (6.23) the β_s factors shift, and for each extra external leg, one therefore gets diagrams which falloff slower by one power of z. Provided that the leading large z falloff cancels for $n > 6$ as it does for $n = 6$, one is lead to expect that under a 123-shift, the NMHV graviton amplitudes behaves as

$$M_n(\hat{1}^-, \hat{2}^-, \hat{3}^-, 4^+, 5^+, \ldots, n^+) \sim \frac{1}{z^{12-n}} \tag{6.58}$$

for large z.[25] We have verified this behavior in explicit numerical work for $n = 5, \ldots, 11$. This is done by calculating the MHV-vertex expansion for each n, testing numerically that the sum of $3(2^{n-3} - 1)$ diagrams is independent of $|X]$. Then another 123-shift with an arbitrary reference spinor is performed on the result for the amplitude, and the leading z falloff is read off from a series expansion as $z \to \infty$. As an extra check we have also calculated M_n for $n = 5, \ldots, 9$ with the recursion relations associated with the 2-line shift $[2,1\rangle$ and numerically tested that the result agrees with the MHV-vertex expansion.

This means that for $n \geq 12$ we must expect the MHV-vertex decomposition to be valid only for the specific choices of $|X]$ which eliminate the $O(1)$-term. It also means that as the number of external legs grow, the spin factors arising from external states other than gravitons will come to dominate the gravity prefactors for large z, and so there will be more bad amplitudes, more very bad amplitudes and also very very bad amplitudes. We expect that these can all be handled by choosing $|X]$ to set the $O(1)$-term to zero.

[25] Other shifts $--+$, $-++$ and $+++$ give asymptotic z^{n-4} behavior.

As a final point, it is worth noting that the n-point NMHV graviton amplitudes continue to be calculable from recursion relations based on 2-line shifts. We have indeed checked numerically for $n = 5, \ldots, 10$ that the two line shifts $[-,-\rangle$, $[-,+\rangle$, $[+,+\rangle$-shifts give $1/z^2$ for large z, while a $[+,-\rangle$-shift gives z^6. This is expected from the general analysis of [31].

Cachazo et al provided in [30] the first proof of the validity of the 2-line recursion relations for graviton amplitudes. Our numerical work confirms their results for 2-line shifts, but it disagrees with their statement that the 3-line shift is valid because it can be obtained by successive 2-line shifts as in (6.57).

7. Discussion and open problems

In this paper we have studied n-point tree amplitudes with general external particles of $\mathcal{N} = 4$ SYM theory and $\mathcal{N} = 8$ supergravity. We have elucidated properties of the generating function proposed for MHV amplitudes in the gauge theory in [14] and extended to the NMHV level in [15], and we have developed similar generating functions for supergravity. The generating function is a simple function of auxiliary Grassmann variables. There is a 1:1 correspondence between particles of the $\mathcal{N} = 4$ and $\mathcal{N} = 8$ theories and Grassmann derivatives, and any desired amplitude is obtained by applying the appropriate product of derivatives to the generating function.

Any n-point MHV amplitude is the product of the "top" n-gluon or n-graviton amplitude times a "spin factor" depending on the external particles. The spin factor of every MHV process in supergravity is a homogeneous function of weight 16 (weight 8 in gauge theory) of the spinors $|i\rangle$ associated with the external particles. We have found a curious and rather perfect analogy between the structure of the spin factors and the structure of holomorphic correlation functions in conformal field theory on the complex plane.

The MHV generating function neatly encodes the full set of spin factors and it allows one to count the number of independent MHV processes. It also clarifies how $\mathcal{N} = 8$ supersymmetry and SU(8) global symmetry of supergravity are implemented in quadratic relations between gauge theory and supergravity amplitudes such as the KLT formulas and the MHV-level formula of [25]. It turns out that, for each permutation in those formulas, the supergravity generating function factors into the product of two gauge theory generating functions.

At the NMHV level the situation is similar, but there is a different generating function for each diagram in the MHV-vertex expansion of the amplitude. Our application to supergravity requires clarification of an important feature of past work. In the MHV-vertex construction, the contribution of each diagram depends on an arbitrary reference spinor $|X]$. In past work it has always been assumed that the full amplitude obtained by summing all diagrams were independent of $|X]$. It was proven in [16] that this property holds for n-gluon NMHV amplitudes, but the argument used does not readily extend to other situations.

The recursion relations obtained from the 3-line shift of [18] provide a precise framework for the MHV-vertex method, and they clarify the role and origin of the reference spinor $|X]$ which determines the shift. The diagrammatic expansion of an amplitude M is valid if the shifted amplitude vanishes at infinity, $M(z, |X]) \to 0$ as $z \to \infty$. If this condition is satisfied for all $|X]$, then the derivation of the recursion relation from Cauchy's theorem ensures that the sum of MHV-vertex diagrams will be independent of $|X]$. If it is not satisfied, the expansion will not produce the right amplitude because the contribution from infinity required by Cauchy's theorem is neglected. Then the sum of diagrams may well depend on $|X]$.

The situation can be made sharper. The 3-line recursion relation is valid if the residue at $z = \infty$ vanishes for *some values* of $|X]$; not necessarily *all values*. The sum of diagrams will produce the correct physical amplitude at precisely those values.

A study of the large z behavior of the individual MHV-vertex diagrams as well as explicit MHV-vertex constructions of many 6-point NMHV amplitudes, indicate that the property $M(z, |X]) \to 0$ as $z \to \infty$ holds for all $|X]$ for general external states in $\mathcal{N} = 4$ SYM theory. But our results also show that there is a contribution from the term at infinity for many supergravity amplitudes. The amplitude calculated from the NMHV generating function is correct provided that $|X]$ is chosen such that the contribution from infinity vanishes.

The appropriate values of $|X]$ for which the MHV-vertex expansion is justified can be determined from a supplementary calculation of the amplitude using the KLT formula. The special values of $|X]$ are roots of a polynomial which precisely expresses the condition that the large z portion of the Cauchy contour integral vanishes. The sum of the diagrammatic expansion is the same at each of these roots and agrees with the value from KLT. This is a pragmatic test of the validity of our approach. Because of the algebraic complexity of NMHV amplitudes in supergravity, our computations are done primarily after input of numerical values for the spinors $|i\rangle$, $|i]$ which contain the information on particle momenta. However, it is clear that the large z polynomial involves only Lorentz invariant spinor brackets $\langle i\, j\rangle$, $[i\, j]$, so that the procedure does not violate Lorentz invariance. Furthermore the key role of the generating function, namely that it encodes the spin factors of all NMHV processes, is preserved.

It is awkward that a supplementary calculation of an amplitude is needed to fix the reference spinor $|X]$ and make the MHV-vertex expansion valid. It would be interesting and useful to find techniques to determine the pole at infinity without a full evaluation of the amplitude.[26]

The generating function approach can be used to simplify the intermediate state helicity sums needed to obtain the integrands of Feynman loop diagrams from products of tree amplitudes by the general cutting techniques widely applied in the work of Bern, Dixon, and Kosower and their collaborators. The MHV level sums carried out in section 4 are really simple, and preliminary calculations of NMHV level sums in the gauge theory are promising. It may be difficult to implement NMHV sums in supergravity because of the necessity to fix $|X]$, as we have discussed above.

For n-graviton amplitudes with $n \geq 5$, an analytic study indicates that there are

[26] We thank P. Benincasa and D. Skinner for discussions of this question.

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individual MHV-vertex diagrams which grow at the rate z^{n-11} under a $---$ shift. (This is the same behavior as in individual terms of the KLT formula.) Subsequent numerical evaluation of the sum of diagrams shows that there is a leading order cancellation, so that the full amplitude grows at the rate z^{n-12}. For general external states, spin factors can contribute extra positive powers z^k with $0 \leq k \leq 8$. The large z behavior of tree amplitudes is related to UV behavior at the 1-loop level [8]. It would be interesting to understand whether the large z growth we have found for 3-line shifts in supergravity may be an omen of future problems with its UV structure.

An aspect of $\mathcal{N} = 8$ supergravity we have not yet discussed is the nonlinearly realized $E(7,7)$ symmetry. Details of the action of $E(7,7)$ are nicely described in the recent paper [33]. The 70 scalar fields of the theory are Goldstone bosons of the spontaneous breaking $E(7,7) \to \mathrm{SU}(8)$. We would like to find the footprint of $E(7,7)$ symmetry in the set of n-point tree amplitudes we have studied. We expected that $E(7,7)$ would reveal itself in the limit of vanishing boson 4-momentum, as in the low energy theorems for soft pion emission obtained long ago by Adler [34] and discussed by Coleman [35]. In pion physics, the low energy limit of a single soft pion is generally non-vanishing and obtained from the sum of Feynman diagrams in which the soft pion is attached to other external lines. Graphs with internal attachment vanish at low energy because of the coupling to the axial current. The soft pion limit is non-zero in tree approximation for the process $\pi + N \to 2\pi + N$ even in a version of the linear σ-model with gradient coupling $\bar{N}\gamma^\mu[\partial_\mu\sigma + i\gamma_5\vec{\tau}\cdot\partial_\mu\vec{\pi}]N$ so that both π and N are massless.

We examined the one-soft-boson limits of our tree amplitudes and found that the limit always vanishes. This was puzzling because the Lagrangian has [36] cubic vertices in which the Goldstone bosons couple to two graviphotons and to a gravitino-graviphotino pair. This leads to diagrams with external line insertions, but their soft limit vanishes when all external particles are on shell. So our results are consistent, but it is still puzzling why soft boson limits are trivial in supergravity but not in pion physics.[27]

Another aspect of our construction, which was only noted in section 3.1, is the tantalizing 1/2 BPS structure of the generating functions for tree level MHV amplitudes. Introducing the 'standard' Grassman variables of $\mathcal{N} = 4$ on-shell superspace $\theta^a_{\dot\alpha}$ and $\bar\theta^\alpha_a$, one can rewrite the MHV generating function as an integral over only a chiral half of superspace since

$$\delta^{(8)}(\sum_i \eta^i_a \lambda^{\dot\alpha}_i) = \int d^8\theta \exp\left(\theta^a_{\dot\alpha}\sum_i \eta^i_a\lambda^{\dot\alpha}_i\right). \tag{7.1}$$

One can go a step further and observe that the ladder of differential operators acting on the auxiliary η's is strikingly reminiscent of the ladder of 'classical' fields generated by the action of (broken) supersymmetry on a self-dual (instanton) configuration. The 8 spinors $\theta^a_{\dot\alpha}$ are the bookkeeper for the fermionic zero-modes. As in supersymmetric instanton calculus, see *e.g.* [37] for a recent review, for a non-zero result one must to 'soak up' the 8 fermionic

[27] It appears that the limit of two soft bosons is closer to the situation in pion physics. There are low energy theorems which reflect the fact that the equal-time commutator of two $E(7,7)$ coset currents lies in the compact SU(8) subalgebra.

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zero modes of chiral $\mathcal{N}=4$ superspace. This selects exactly the set of 15 types of MHV related amplitudes. It is tempting to conjecture that the functional integral representation of MHV amplitudes is 'dominated' by classical self-dual configurations of the gauge field whose precise form depends on the boundary conditions dictated by the choice of external momenta [38, 39]. This is further supported by the recent results of [40, 41] where the $\mathcal{N}=4$ multiplet is packaged into a scalar light-cone (LC) superfield that only depends on half the LC superspace θ_{LC}'s. For related work in the supertwistor formulation, see for example [42]. Similar considerations may apply to $\mathcal{N}=8$ supergravity, where the recent LC $\mathcal{N}=8$ superspace approach of [43] supports the 1/2 BPS structure of MHV amplitudes and sheds some light on the role of $E(7,7)$.

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Acknowledgments

We thank Z. Bern and L. Dixon for much advice and encouragement, and we acknowledge illuminating conversations with S. Adler, I. Antoniadis, E. Bjerrum-Bohr, L. Brink, S. Ferrara, G. Gibbons, M. B. Green, D. Kosower, S. Kovacs, R. Kallosh, M. Mangano, E. Sokatchev, S. Stieberger, and P. Vanhove.

Special thanks are due to P. Benincasa for pointing out the incorrect statements concerning large z behavior in the first version of this paper. His observation broadens the applicability of our methods.

DZF is grateful for financial support from TH-Division at CERN, the Stanford Institute for Theoretical Physics, and the Centre for Theoretical Cosmology of Cambridge University during his sabbatical in 2007-2008. The research of DZF is also supported by NSF grant PHY-0600465.

The work of MB has been supported in part by the European Community Human Potential Program under contract MRTN-CT-2004-512194, by the INTAS grant 03-516346, by MIUR-COFIN 2003-023852, and by NATO PST.CLG.978785. Preliminary results of this work were presented by MB at the conference "String Phenomenology and Dynamical Vacuum Selection", Liverpool, UK, March 27-29 2008. MB would like to thank the organizers for the kind invitation and the stimulating atmosphere.

HE is supported by a Pappalardo Fellowship in Physics at MIT and in part by the US Department of Energy through cooperative research agreement DE-FG0205ER41360

A. Conventions

Our notation for the spinor helicity formalism is largely inspired by [22, 44], but we use different conventions which are summarized in the following.

A.1 Spinor helicity formalism

We work with a mostly-plus metric, $\eta_{\mu\nu} = \mathrm{diag}(-1,+1,+1,+1)$. Gamma-matrices are defined as

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix}, \qquad \{\gamma^\mu,\gamma^\nu\} = 2\eta^{\mu\nu}, \qquad \gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{A.1}$$

with $\sigma^\mu = (1, \sigma^i)$, $\bar{\sigma}^\mu = (-1, \sigma^i)$ and σ^i the standard Pauli matrices.[28]

Positive and negative helicity solutions of the massless Dirac equation, $\gamma \cdot p\, u_s(p) = 0$, are written in terms of commuting 2-component spinors $\tilde{\lambda}$ and λ defined as

$$u_-(p) = \begin{pmatrix} \lambda_\alpha \\ 0 \end{pmatrix}, \qquad u_+(p) = \begin{pmatrix} 0 \\ \tilde{\lambda}^{\dot{\alpha}} \end{pmatrix}. \tag{A.2}$$

Projectors $P_\pm = \frac{1}{2}(1 \pm \gamma_5)$ then act as $P_\pm u_\pm(p) = u_\pm(p)$ and $P_\pm u_\mp(p) = 0$. With the adjoint of a Dirac spinor Ψ defined as

$$\bar{\Psi} \equiv -i \Psi^\dagger \gamma^0 , \tag{A.3}$$

we have

$$\overline{u}_-(p) = (0, -i\,\tilde{\lambda}_{\dot{\alpha}}) , \qquad \overline{u}_+(p) = (i\,\lambda^\alpha, 0) , \tag{A.4}$$

where $\lambda^\alpha = (\tilde{\lambda}^{\dot{\alpha}})^*$ and $\tilde{\lambda}_{\dot{\alpha}} = (\lambda_\alpha)^*$. Note that $\lambda^\alpha = \epsilon^{\alpha\beta}\lambda_\beta$ and $\tilde{\lambda}^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}\tilde{\lambda}_{\dot{\beta}}$.

Defining $p_{\alpha\dot{\beta}} = p_\mu\,(\sigma^\mu)_{\alpha\dot{\beta}}$ and $p^{\dot{\alpha}\beta} = p_\mu\,(\bar{\sigma}^\mu)^{\dot{\alpha}\beta}$, the massless Dirac equation can be written

$$p^{\dot{\alpha}\beta}\lambda_\beta = 0 , \qquad \tilde{\lambda}_{\dot{\alpha}}\, p^{\dot{\alpha}\beta} = 0 \qquad p_{\alpha\dot{\beta}}\tilde{\lambda}^{\dot{\beta}} = 0 , \qquad \lambda^\alpha\, p_{\alpha\dot{\beta}} = 0 . \tag{A.5}$$

One can show that

$$\lambda_\alpha \tilde{\lambda}_{\dot{\beta}} = -p_{\alpha\dot{\beta}} , \qquad \lambda^\alpha \tilde{\lambda}^{\dot{\beta}} = +p^{\dot{\beta}\alpha} . \tag{A.6}$$

We now introduce the bra-ket notation which is used heavily throughout the paper. Define

$$|p] = u_-(p) = \begin{pmatrix} \lambda_\alpha \\ 0 \end{pmatrix}, \qquad |p\rangle = u_+(p) = \begin{pmatrix} 0 \\ \tilde{\lambda}^{\dot{\alpha}} \end{pmatrix}, \tag{A.7}$$

$$\langle p| = i\,\bar{u}_-(p) = \big(0,\ \tilde{\lambda}_{\dot{\alpha}}\big) , \qquad [p| = -i\,\bar{u}_+(p) = \big(\lambda^\alpha,\ 0\big) , \tag{A.8}$$

It then follows from (A.6) that

$$-p_\mu \gamma^\mu = |p]\langle p| - |p\rangle [p| . \tag{A.9}$$

Spinor products are defined as

$$\langle p\, q\rangle = \tilde{\lambda}_{p\,\dot{\alpha}}\tilde{\lambda}_q^{\dot{\alpha}} , \qquad [p\, q] = \lambda_p^\alpha \lambda_{q\,\alpha} , \tag{A.10}$$

and they are related to the dot-product of the momenta by

$$\langle p\, q\rangle\, [p\, q] = 2\, p \cdot q = -s_{pq} , \tag{A.11}$$

where the Mandelstam variables are $s_{pq} = -(p+q)^2$. For real momenta, the spinor products satisfy $[p\, q]^* = \langle q\, p\rangle$, so that up to phases $[p\, q] \sim \langle p\, q\rangle \sim \sqrt{2\, p \cdot q}$. In applications we often

[28] Note that $(\bar{\sigma}^\mu)^{\dot{\alpha}\beta} = -\epsilon^{\beta\gamma}\epsilon^{\dot{\alpha}\dot{\delta}}(\sigma^\mu)_{\gamma\dot{\delta}}$. We use $\epsilon^{12} = \epsilon_{12} = 1$.

use complex momenta in which case angle and square brackets ($\tilde{\lambda}$ and λ) will not be complex conjugates, but independent. We remark on the properties of the angle and square spinors under analytic continuation $p \to -p$. In our conventions, $|-p\rangle = -|p\rangle$ and $|-p] = +|p]$.

It is convenient to define "angle-square brackets" $\langle i|P|j]$ as

$$\langle i|P|j] = \sum_{k=1}^{m} \langle i\, k\rangle\, [k\, j] \quad \text{for} \quad P = \sum_{k=1}^{m} p_{i_k}\,. \tag{A.12}$$

In the spinor helicity formalism polarization vectors can be written as

$$\epsilon_+^\mu(p;q) = -\frac{[q|\gamma^\mu|p\rangle}{\sqrt{2}\,[q\,p]}\,, \qquad \epsilon_-^\mu(p;q) = \frac{\langle q|\gamma^\mu|p]}{\sqrt{2}\,\langle q\,p\rangle}\,. \tag{A.13}$$

One can show[29] that the polarization vectors are related by complex conjugation and satisfy the orthogonality relations

$$\left(\epsilon_\pm^\mu(p)\right)^* = -\epsilon_\mp^\mu(p)\,, \qquad \epsilon_s^\mu(p)^* \epsilon_{\mu\, s'}(p) = \delta_{ss'}\,. \tag{A.14}$$

A.2 Explicit representation

Take the momentum to be

$$p^\mu = (E,\ E\,\sin\theta\cos\phi,\ E\,\sin\theta\sin\phi,\ E\,\cos\theta)\,. \tag{A.15}$$

Then

$$p_{\alpha\dot{\beta}} = -2E\begin{pmatrix} s^2 & -c\,s\,e^{-i\phi} \\ -c\,s\,e^{i\phi} & c^2 \end{pmatrix}\,, \qquad p^{\dot{\alpha}\beta} = 2E\begin{pmatrix} c^2 & c\,s\,e^{-i\phi} \\ c\,s\,e^{i\phi} & s^2 \end{pmatrix}\,, \tag{A.16}$$

where we use $s = \sin\frac{\theta}{2}$ and $c = \cos\frac{\theta}{2}$. It is straightforward to show that the two-component vectors

$$\lambda_\alpha = \sqrt{2E}\begin{pmatrix} -s\,e^{-i\phi/2} \\ c\,e^{i\phi/2} \end{pmatrix}\,, \qquad \tilde{\lambda}_{\dot{\alpha}} = \sqrt{2E}\left(-\,s\,e^{i\phi/2},\ c\,e^{-i\phi/2}\right)\,, \tag{A.17}$$

$$\tilde{\lambda}^{\dot{\alpha}} = \sqrt{2E}\begin{pmatrix} c\,e^{-i\phi/2} \\ s\,e^{i\phi/2} \end{pmatrix}\,, \qquad \lambda^\alpha = \sqrt{2E}\left(c\,e^{i\phi/2},\ s\,e^{-i\phi/2}\right) \tag{A.18}$$

solve the massless Dirac equation in the form (A.5).

With p_μ given by (A.15), we write the positive and negative helicity vectors

$$\epsilon_\pm^\mu(p) = \mp\frac{1}{\sqrt{2}}\Big(0,\ \cos\theta\,\cos\phi \mp i\sin\phi,\ \pm i\cos\phi + \cos\theta\,\sin\phi,\ -\sin\theta\Big)\,. \tag{A.19}$$

These clearly satisfy (A.14). It can be shown that the expressions (A.13) reproduce the polarization vectors (A.19) for an appropriate choice of reference momentum q.

[29] It is useful to note the following properties: $\langle p|P|q] = P_\mu\langle p|\gamma^\mu|q]$, $[q|\gamma^\mu|p\rangle = -\langle p|\gamma^\mu|q]$, $[q|\gamma^\mu|p\rangle^* = [p|\gamma^\mu|q\rangle$, and $[p_1|\gamma^\mu|p_2\rangle\langle p_3|\gamma_\mu p_4] = 2[p_1 p_4]\langle p_3 p_2\rangle$.

A.3 Majorana spinors

A Majorana spinor satisfies the condition

$$\psi \;=\; B^{-1}\psi^* , \qquad B \;=\; \gamma^0\gamma^1\gamma^3 \;=\; \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} . \tag{A.20}$$

In the γ-matrix representation (A.1) this means that

$$\psi \;=\; \begin{pmatrix} \psi_\alpha \\ \tilde\psi^{\dot\alpha} \end{pmatrix} \qquad \text{with} \qquad \psi_\alpha \;=\; \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \qquad \text{and} \qquad \tilde\psi^{\dot\alpha} \;=\; \begin{pmatrix} \psi_2^* \\ -\psi_1^* \end{pmatrix} . \tag{A.21}$$

It follows from this and $\tilde\psi^{\dot\alpha} = \epsilon^{\dot\alpha\dot\beta}\tilde\psi_{\dot\beta}$ that $\tilde\psi_{\dot\alpha} = (\psi_\alpha)^*$.

Let ε and $\mathcal{Q}$ be Majorana spinors. Then

$$\bar\varepsilon\,\mathcal{Q} \;=\; -\,i\big(\epsilon_1^*Q_2^* - \epsilon_2^*Q_1^* - \epsilon_2 Q_1 + \epsilon_1 Q_2\big) \;=\; -\,i\big(\tilde\epsilon_{\dot\alpha}\tilde Q^{\dot\alpha} - \epsilon^\alpha Q_\alpha\big) \;\equiv\; -\,i\big(\tilde Q + Q\big) \tag{A.22}$$

If $\delta_\epsilon A$ is the supersymmetry transformation of the field A, then the susy generators act as

$$\delta_\epsilon A \;=\; i\big[\bar\varepsilon\,\mathcal{Q}, A\big] \;=\; \big[Q + \tilde Q, A\big] . \tag{A.23}$$

Including labels $a, b, \dots = 1, \dots, \mathcal{N}$, the generators Q^a and $\tilde Q_b$ satisfy the extended supersymmetry algebra

$$\big[[Q^a, \tilde Q_b], A\big] \;=\; \delta^a_b \langle\epsilon_2\, p\rangle[p\,\epsilon_1]\, A , \qquad \big[[Q^a, Q^b], A\big] \;=\; 0 , \qquad \big[[\tilde Q_a, \tilde Q_b], A\big] \;=\; 0 , \tag{A.24}$$

for distinct susy parameters $\epsilon_{1,2}$ and $\tilde\epsilon_{1,2}$.

B. Solution of the $\mathcal{N}=1$ SUSY Ward identities for 6-point NMHV amplitudes

We apply spinor-helicity methods to obtain the solution to the SUSY Ward identities for 6-point NMHV amplitudes in an $\mathcal{N}=1$ theory originally found in [21].[30] The solution is valid for complex momenta in an arbitrary Lorentz frame and reduces to the solution of [21] for real momenta in the center-of-mass frame. We hope that the new method will be useful for solving the NMHV Ward identities of extended supersymmetry.

Let $b^\pm$ and $f^\pm$ denote the annihilators of the bosonic and fermionic states of an $\mathcal{N}=1$ supersymmetric theory. There are 20 independent 6-point NMHV amplitudes for which we introduce the notation:

$$G = \sigma_b \left\langle b^+b^+b^+b^-b^-b^-\right\rangle , \qquad G_{i,I} = (-)^{i+I}\sigma_b \left\langle f^+b_i^+f^+f^-b_I^-f^-\right\rangle , \tag{B.1}$$

$$F = \sigma_f \left\langle f^+f^+f^+f^-f^-f^-\right\rangle , \qquad F_{i,I} = \sigma_f \left\langle b^+f_i^+b^+b^-f_I^-b^-\right\rangle . \tag{B.2}$$

[30] See also [45].

The momentum (and position) labels i, j, k run over $1, 2, 3$ while $I, J, K = 4, 5, 6$. The subscript i on b_i^+ means that the particle is in position i with momentum p_i, etc. For example, $G_{1,6} = -\sigma_b \big\langle b^+(1)\, f^+(2)\, f^+(3)\, f^-(4)\, f^-(5)\, b^-(6)\big\rangle$.

The supersymmetric Ward identities can be solved to express the 18 amplitudes $G_{i,I}$ and $F_{i,I}$ in terms of the purely bosonic and fermionic amplitudes G and F. The result is

$$F_{i,I} = \Delta^{-1}\big(\epsilon_{ijk}\langle jk\rangle\, \epsilon_{IJK}[JK]\, F + 4\langle Ij\rangle[ij]\, G\big)\,, \tag{B.3}$$

$$G_{i,I} = \Delta^{-1}\big(\epsilon_{ijk}[jk]\, \epsilon_{IJK}\langle JK\rangle\, G + 4\langle iJ\rangle[IJ]\, F\big)\,, \tag{B.4}$$

where

$$\Delta \;=\; -\,2\langle ij\rangle\,[ij] \;=\; -\,2\langle IJ\rangle\,[IJ]\,, \tag{B.5}$$

and repeated indices are summed. The remainder of this appendix is devoted to the proof of (B.3)–(B.4).

The commutator relations of the $\mathcal{N} = 1$ SUSY generator $\tilde{Q}$ with the annihilators is

$$\begin{aligned} &\big[\tilde{Q}, b^+(p)\big] = \sigma_b\langle\epsilon\, p\rangle\, f^+(p)\,, && \big[\tilde{Q}, f^+(p)\big] = 0\,, \\ &\big[\tilde{Q}, b^-(p)\big] = 0\,, && \big[\tilde{Q}, f^-(p)\big] = \sigma_f\langle\epsilon\, p\rangle\, b^-(p)\,. \end{aligned} \tag{B.6}$$

The phases $\sigma_{b,f} = \pm 1$ depend on which $\mathcal{N} = 1$ multiplet is considered. Similar relations exist for the generator Q which raises the helicity by $1/2$.

The SUSY Ward identities from the $\tilde{Q}$ commutator relations can be written compactly as

$$\begin{aligned} \langle\epsilon\, i\rangle\, F_{i,I} + \langle\epsilon\, I\rangle\, G \;&=\; 0\,, \\ \langle\epsilon\, i\rangle\, F + \langle\epsilon\, I\rangle\, G_{i,I} \;&=\; 0\,, \\ \epsilon_{ijk}\langle\epsilon\, j\rangle\, G_{k,I} + \epsilon_{IJK}\langle\epsilon\, J\rangle\, F_{i,K} \;&=\; 0\,. \end{aligned} \tag{B.7}$$

We will also need a subset of the Ward identities obtained from the conjugate Q Ward identities. In notation that should be obvious, these read

$$\sigma\,[\epsilon\, i]\, G - [\epsilon\, I]F_{i,I} \;=\; 0\,, \qquad \sigma\,[\epsilon\, I]\, F - [\epsilon\, i]G_{i,I} \;=\; 0\,, \tag{B.8}$$

with $\sigma = \pm 1$ resulting from the choice of phases in the algebra of Q with the annihilators.

We start with the solution ansatz

$$F_{i,I} \;=\; M_{i,I}\, G + N_{i,I}\, F\,, \qquad G_{i,I} \;=\; K_{i,I}\, G + L_{i,I}\, F\,, \tag{B.9}$$

where M, N, K, L are 3×3 matrices. The Ward identities (B.7) split into two sets of equations, one set for the matrices N and L,

$$\langle\epsilon\, i\rangle N_{i,I} \;=\; 0\,, \qquad \langle\epsilon\, I\rangle L_{i,I} \;=\; -\,\langle\epsilon\, i\rangle\,, \qquad \epsilon_{ijk}\,\langle\epsilon\, j\rangle L_{k,I} + \epsilon_{IJK}\,\langle\epsilon\, J\rangle N_{i,K} \;=\; 0\,, \tag{B.10}$$

and another for M and K

$$\langle\epsilon\, I\rangle K_{i,I} \;=\; 0\,, \qquad \langle\epsilon\, i\rangle M_{i,I} \;=\; -\,\langle\epsilon\, I\rangle\,, \qquad \epsilon_{ijk}\,\langle\epsilon\, j\rangle K_{k,I} + \epsilon_{IJK}\,\langle\epsilon\, J\rangle M_{i,K} \;=\; 0\,. \tag{B.11}$$

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In addition equation (B.8) gives (among other relations)

$$[\epsilon\, I]N_{k,I} \;=\; 0\,, \qquad [\epsilon\, k]K_{k,I} \;=\; 0\,. \tag{B.12}$$

Due to the separation of the constraints, we will focus our attention on the equations for N and L; the system of K, M equations is identical and is treated the same way.

The equation $\langle\epsilon\, i\rangle N_{i,I} = 0$ is simply solved by $N_{i,I} = \epsilon_{ijk}\langle j\, k\rangle n_I$ for any vector n_I. This follows from the Schouten identity. Next, $[\epsilon\, I]N_{k,I} = 0$ is solved by $n_I = \epsilon_{IJK}[J\, K]\Delta^{-1}$ for some general function Δ to be determined. Hence

$$N_{i,I} \;=\; \Delta^{-1}\epsilon_{ijk}\langle j\, k\rangle\epsilon_{IJK}[J\, K]\,. \tag{B.13}$$

Using the standard identity $\epsilon_{IJK}\,\epsilon_{KLM} = (\delta_{IL}\delta_{JM} - \delta_{IM}\delta_{JL})$, the third equation of (B.10) then gives

$$\frac{2}{\Delta}\epsilon_{ijk}\langle j\, k\rangle\langle\epsilon\, J\rangle[I\, J] \;=\; -\,\epsilon_{ijk}\langle\epsilon\, j\rangle L_{k,I}\,. \tag{B.14}$$

Multiplying both sides with ϵ^{ilm} and summing over i we then find

$$\frac{4}{\Delta}\langle\epsilon\, J\rangle[IJ]\langle l\, m\rangle \;=\; -\,\langle\epsilon\, l\rangle L_{m,I} + \langle\epsilon\, m\rangle L_{l,I}\,. \tag{B.15}$$

Choosing $\langle\epsilon| = \langle l|$ (no sum on l) provides the solution for L; it is

$$L_{l,I} \;=\; \frac{4}{\Delta}\langle l\, J\rangle[IJ]\,. \tag{B.16}$$

The only task left now is to determine the scalar function Δ. This is easily done as follows. Multiply (B.15) by $\langle\epsilon'\, I\rangle$ for some arbitrary spinor ϵ'. Summing over I and using $\langle\epsilon'\, I\rangle L_{i,I} \;=\; -\,\langle\epsilon'\, i\rangle$ we obtain

$$\frac{4}{\Delta}\langle\epsilon'\, I\rangle\langle\epsilon\, J\rangle[IJ]\langle l\, m\rangle \;=\; \langle\epsilon\, l\rangle\langle\epsilon'\, m\rangle - \langle\epsilon\, m\rangle\langle\epsilon'\, l\rangle \;=\; -\,\langle\epsilon\,\epsilon'\rangle\langle m\, l\rangle\,. \tag{B.17}$$

Antisymmetrization of IJ on the l.h.s. of (B.17) gives (by Schouten)

$$\frac{4}{\Delta}\langle\epsilon'\, I\rangle\langle\epsilon\, J\rangle[IJ] \;=\; \frac{2}{\Delta}\Big(\langle\epsilon'\, I\rangle\langle\epsilon\, J\rangle - \langle\epsilon'\, J\rangle\langle\epsilon\, I\rangle\Big)[IJ] \;=\; -\,\frac{2}{\Delta}\langle\epsilon\,\epsilon'\rangle\langle IJ\rangle[IJ]\,. \tag{B.18}$$

Thus the factors of $\langle\epsilon\,\epsilon'\rangle$ can be eliminated and we conclude from (B.17) and (B.18) that

$$\Delta \;=\; -\,2\langle IJ\rangle[IJ]\,. \tag{B.19}$$

Note that momentum conservation implies that

$$\Delta = -2\langle IJ\rangle[IJ] = -2\sum_{I,J}(p_I + p_J)^2 = -2\sum_{i,j}(p_i + p_j)^2 = -2\langle ij\rangle[ij]\,. \tag{B.20}$$

This completes the solution for N and L. Since the system of equations for K and M is identical, we have proven that (B.3)–(B.4) solve the $\mathcal{N} = 1$ SUSY Ward identities.

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